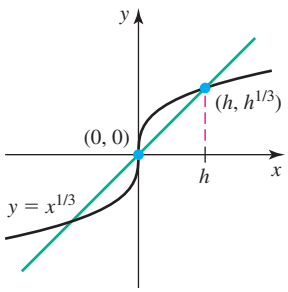
46–47. Steep secant lines

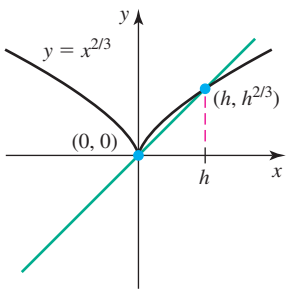
a. *Given the graph of f in the following figures, find the slope of the secant line that passes through $(0, 0)$ and $(h, f(h))$ in terms of h, for $h > 0$ and $h < 0$.*

b. *Evaluate the limit of the slope of the secant line found in part (a) as $h \to 0^+$ and $h \to 0^-$. What does this tell you about the line tangent to the curve at $(0, 0)$?*

46. $f(x) = x^{1/3}$



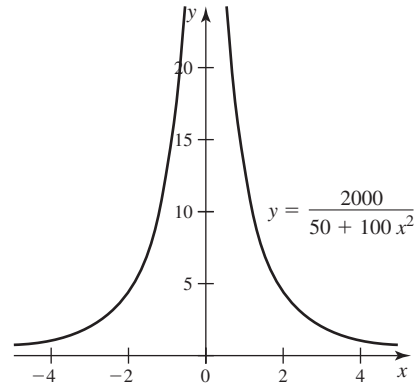
47. $f(x) = x^{2/3}$



T **48. Care with graphing** The figure shows the graph of the function $f(x) = \dfrac{2000}{50 + 100x^2}$ graphed in the window $[-4, 4] \times [0, 20]$.

a. Evaluate $\lim\limits_{x \to 0^+} f(x)$, $\lim\limits_{x \to 0^-} f(x)$, and $\lim\limits_{x \to 0} f(x)$.

b. Create a graph that gives a more complete representation of f.



Technology Exercises

T **49–56. Asymptotes** *Use analytical methods and/or a graphing utility to identify the vertical asymptotes (if any) of the following functions.*

49. $f(x) = \dfrac{x^2 - 3x + 2}{x^{10} - x^9}$

50. $g(x) = 2 - \ln x^2$

51. $h(x) = \dfrac{e^x}{(x + 1)^3}$

52. $p(x) = \sec\left(\dfrac{\pi x}{2}\right)$, for $|x| < 2$

53. $g(\theta) = \tan\left(\dfrac{\pi\theta}{10}\right)$

54. $q(s) = \dfrac{\pi}{s - \sin s}$

55. $f(x) = \dfrac{1}{\sqrt{x}\sec x}$

56. $g(x) = e^{1/x}$

T **57. Can a graph intersect a vertical asymptote?** A common misconception is that the graph of a function never intersects its vertical asymptotes. Let

$$f(x) = \begin{cases} \dfrac{4}{x - 1} & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1. \end{cases}$$

Explain why $x = 1$ is a vertical asymptote of the graph of f and show that the graph of f intersects the line $x = 1$.

QUICK CHECK ANSWERS

1. Answers will vary, but all graphs should have a vertical asymptote at $x = 2$. **2.** $-\infty$; ∞ **3.** As $x \to -4^+$, $x < 0$ and $(x + 4) > 0$, so $x(x + 4) \to 0$ through negative values. **4.** $\lim\limits_{x \to 2} \dfrac{(x - 1)(x - 2)}{x - 2} = \lim\limits_{x \to 2} (x - 1) = 1$, which is not an infinite limit, so $x = 2$ is not a vertical asymptote. ◀

2.5 Limits at Infinity

Limits at infinity—as opposed to infinite limits—occur when the independent variable becomes large in magnitude. For this reason, limits at infinity determine what is called the *end behavior* of a function. An application of these limits is to determine whether a system (such as an ecosystem or a large oscillating structure) reaches a steady state as time increases.

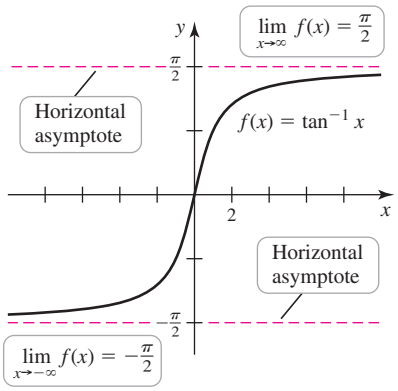


FIGURE 2.30

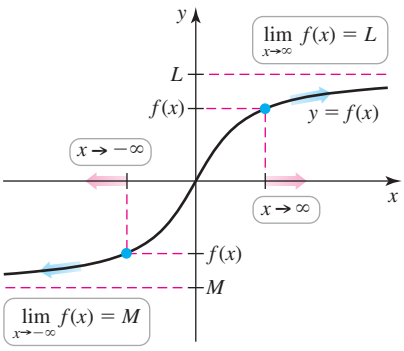


FIGURE 2.31

Limits at Infinity and Horizontal Asymptotes

Consider the function $f(x) = \tan^{-1} x$, whose domain is $(-\infty, \infty)$ (Figure 2.30). As x becomes arbitrarily large (denoted $x \to \infty$), $f(x)$ approaches $\pi/2$, and as x becomes arbitrarily large in magnitude and negative (denoted $x \to -\infty$), $f(x)$ approaches $-\pi/2$. These limits are expressed as

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \quad \text{and} \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}.$$

The graph of f approaches the horizontal line $y = \pi/2$ as $x \to \infty$, and it approaches the horizontal line $y = -\pi/2$ as $x \to -\infty$. These lines are called *horizontal asymptotes*.

DEFINITION Limits at Infinity and Horizontal Asymptotes

If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write

$$\lim_{x\to\infty} f(x) = L.$$

We say the limit of $f(x)$ as x approaches infinity is L. In this case, the line $y = L$ is a **horizontal asymptote** of f (Figure 2.31). The limit at negative infinity, $\lim_{x\to-\infty} f(x) = M$, is defined analogously. When the limit exists, the horizontal asymptote is $y = M$.

QUICK CHECK 1 Evaluate $x/(x+1)$ for $x = 10$, 100, and 1000. What is $\lim_{x\to\infty} \frac{x}{x+1}$?

EXAMPLE 1 Limits at infinity

Evaluate the following limits.

a. $\lim_{x\to-\infty} \left(2 + \frac{10}{x^2}\right)$ **b.** $\lim_{x\to\infty} \left(3 + \frac{3 \sin x}{\sqrt{x}}\right)$

SOLUTION

a. As x becomes large and negative, x^2 becomes large and positive; in turn, $10/x^2$ approaches 0. By the limit laws of Theorem 2.3,

$$\lim_{x\to-\infty} \left(2 + \frac{10}{x^2}\right) = \underbrace{\lim_{x\to-\infty} 2}_{\text{equals } 2} + \underbrace{\lim_{x\to-\infty} \left(\frac{10}{x^2}\right)}_{\text{equals } 0} = 2 + 0 = 2.$$

Notice that $\lim_{x\to\infty} \left(2 + \frac{10}{x^2}\right)$ is also equal to 2. Therefore, the graph of $y = 2 + 10/x^2$ approaches the horizontal asymptote $y = 2$ as $x \to \infty$ and as $x \to -\infty$ (Figure 2.32).

b. The numerator of $\sin x/\sqrt{x}$ is bounded between -1 and 1; therefore, for $x > 0$,

$$-\frac{1}{\sqrt{x}} \le \frac{\sin x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}.$$

As $x \to \infty$, $\sqrt{x}$ becomes arbitrarily large, which means that

$$\lim_{x\to\infty} \frac{-1}{\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0.$$

It follows by the Squeeze Theorem (Theorem 2.5) that $\lim_{x\to\infty} \frac{\sin x}{\sqrt{x}} = 0$.

The limit laws of Theorem 2.3 and the Squeeze Theorem apply if $x \to a$ is replaced with $x \to \infty$ or $x \to -\infty$.

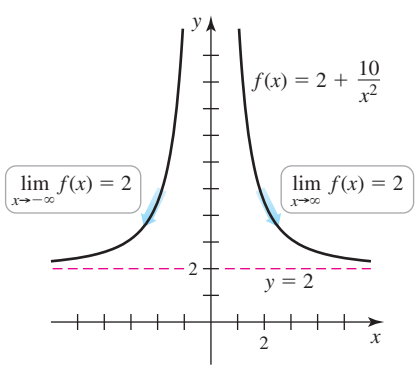


FIGURE 2.32

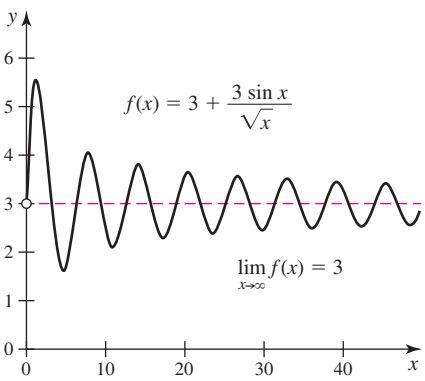


FIGURE 2.33

Using the limit laws of Theorem 2.3,

$$\lim_{x\to\infty}\left(3 + \frac{3 \sin x}{\sqrt{x}}\right) = \underbrace{\lim_{x\to\infty} 3}_{\text{equals } 3} + 3 \underbrace{\lim_{x\to\infty}\left(\frac{\sin x}{\sqrt{x}}\right)}_{\text{equals } 0} = 3.$$

The graph of $y = 3 + \frac{3 \sin x}{\sqrt{x}}$ approaches the horizontal asymptote $y = 3$ as x becomes large (Figure 2.33). Note that the curve intersects its asymptote infinitely many times. *Related Exercises 9–14*◀

Infinite Limits at Infinity

It is possible for a limit to be *both* an infinite limit and a limit at infinity. This type of limit occurs if $f(x)$ becomes arbitrarily large in magnitude as x becomes arbitrarily large in magnitude. Such a limit is called an *infinite limit at infinity* and is illustrated by the function $f(x) = x^3$ (Figure 2.34).

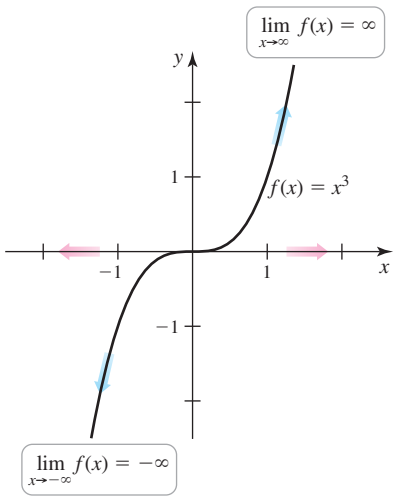


FIGURE 2.34

DEFINITION Infinite Limits at Infinity

If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write

$$\lim_{x\to\infty} f(x) = \infty.$$

The limits $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$ are defined similarly.

Infinite limits at infinity tell us about the behavior of polynomials for large-magnitude values of x. First, consider power functions $f(x) = x^n$, where n is a positive integer. Figure 2.35 shows that when n is even, $\lim_{x\to\pm\infty} x^n = \infty$, and when n is odd, $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$.

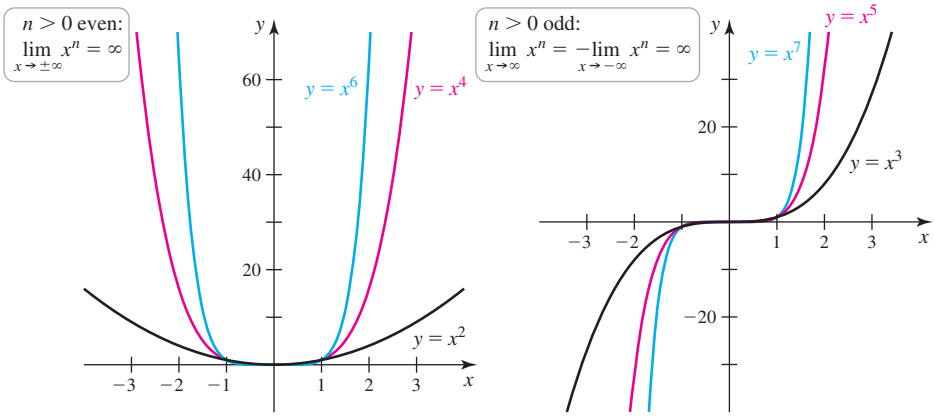


FIGURE 2.35

It follows that reciprocals of power functions $f(x) = 1/x^n = x^{-n}$, where n is a positive integer, behave as follows:

$$\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to\infty} x^{-n} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^n} = \lim_{x\to-\infty} x^{-n} = 0.$$

QUICK CHECK 2 Describe the behavior of $p(x) = -3x^3$ as $x \to \infty$ and as $x \to -\infty$.

From here, it is a short step to finding the behavior of any polynomial as $x \to \pm\infty$. Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$. We now write p in the equivalent form

$$p(x) = x^n\left(a_n + \underbrace{\frac{a_{n-1}}{x}}_{\to 0} + \underbrace{\frac{a_{n-2}}{x^2}}_{\to 0} + \cdots + \underbrace{\frac{a_0}{x^n}}_{\to 0}\right).$$

Notice that as x becomes large in magnitude, all the terms in p except the first term approach zero. Therefore, as $x \to \pm\infty$, we see that $p(x) \approx a_nx^n$. This means that as $x \to \pm\infty$, the behavior of p is determined by the term a_nx^n with the highest power of x.

THEOREM 2.6 Limits at Infinity of Powers and Polynomials
Let n be a positive integer and let p be the polynomial
$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, where $a_n \neq 0$.

1. $\lim_{x\to\pm\infty} x^n = \infty$ when n is even.
2. $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$ when n is odd.
3. $\lim_{x\to\pm\infty} \frac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0$.
4. $\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_nx^n = \infty$ or $-\infty$, depending on the degree of the polynomial and the sign of the leading coefficient a_n.

EXAMPLE 2 Limits at infinity Evaluate the limits as $x \to \pm\infty$ of the following functions.

a. $p(x) = 3x^4 - 6x^2 + x - 10$ **b.** $q(x) = -2x^3 + 3x^2 - 12$

SOLUTION

a. We use the fact that the limit is determined by the behavior of the leading term:

$$\lim_{x\to\infty}(3x^4 - 6x^2 + x - 10) = \lim_{x\to\infty} 3\underbrace{x^4}_{\to\infty} = \infty.$$

Similarly,

$$\lim_{x\to-\infty}(3x^4 - 6x^2 + x - 10) = \lim_{x\to-\infty} 3\underbrace{x^4}_{\to\infty} = \infty.$$

b. Noting that the leading coefficient is negative, we have

$$\lim_{x\to\infty}(-2x^3 + 3x^2 - 12) = \lim_{x\to\infty}(-2\underbrace{x^3}_{\to\infty}) = -\infty$$

$$\lim_{x\to-\infty}(-2x^3 + 3x^2 - 12) = \lim_{x\to-\infty}(-2\underbrace{x^3}_{\to-\infty}) = \infty.$$

Related Exercises 15–24

End Behavior

The behavior of polynomials as $x \to \pm\infty$ is an example of what is often called *end behavior*. Having treated polynomials, we now turn to the end behavior of rational, algebraic, and transcendental functions.

EXAMPLE 3 **End behavior of rational functions** Determine the end behavior for the following rational functions.

a. $f(x) = \dfrac{3x + 2}{x^2 - 1}$ **b.** $g(x) = \dfrac{40x^4 + 4x^2 - 1}{10x^4 + 8x^2 + 1}$ **c.** $h(x) = \dfrac{x^3 - 2x + 1}{2x + 4}$

SOLUTION

a. An effective approach for evaluating limits of rational functions at infinity is to divide both the numerator and denominator by x^n, where n is the largest power appearing in the denominator. This strategy forces the terms corresponding to lower powers of x to approach 0 in the limit. In this case, we divide by x^2:

$$\lim_{x\to\infty} \frac{3x + 2}{x^2 - 1} = \lim_{x\to\infty} \frac{\dfrac{3x + 2}{x^2}}{\dfrac{x^2 - 1}{x^2}} = \lim_{x\to\infty} \frac{\overbrace{\dfrac{3}{x} + \dfrac{2}{x^2}}^{\text{approaches } 0}}{1 - \underbrace{\dfrac{1}{x^2}}_{\text{approaches } 0}} = \frac{0}{1} = 0.$$

Recall that the *degree* of a polynomial is the highest power of x that appears.

A similar calculation gives $\displaystyle\lim_{x\to-\infty} \frac{3x + 2}{x^2 - 1} = 0$, and thus the graph of f has the horizontal asymptote $y = 0$. You should confirm that the zeros of the denominator are -1 and 1, which correspond to vertical asymptotes (Figure 2.36). In this example, the degree of the polynomial in the numerator is *less than* the degree of the polynomial in the denominator.

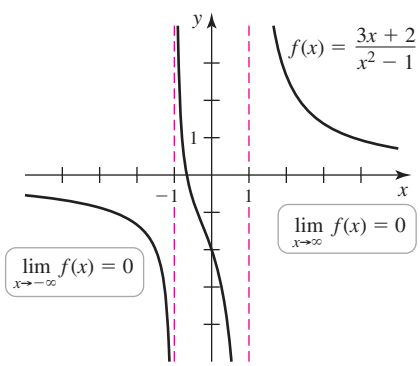


FIGURE 2.36

b. Again we divide both the numerator and denominator by the largest power appearing in the denominator, which is x^4:

$$\lim_{x\to\infty} \frac{40x^4 + 4x^2 - 1}{10x^4 + 8x^2 + 1} = \lim_{x\to\infty} \frac{\dfrac{40x^4}{x^4} + \dfrac{4x^2}{x^4} - \dfrac{1}{x^4}}{\dfrac{10x^4}{x^4} + \dfrac{8x^2}{x^4} + \dfrac{1}{x^4}} \qquad \text{Divide the numerator and denominator by } x^4.$$

$$= \lim_{x\to\infty} \frac{40 + \overbrace{\dfrac{4}{x^2}}^{\text{approaches } 0} - \overbrace{\dfrac{1}{x^4}}^{\text{approaches } 0}}{10 + \underbrace{\dfrac{8}{x^2}}_{\text{approaches } 0} + \underbrace{\dfrac{1}{x^4}}_{\text{approaches } 0}} \qquad \text{Simplify.}$$

$$= \frac{40 + 0 + 0}{10 + 0 + 0} = 4. \qquad \text{Evaluate limits.}$$

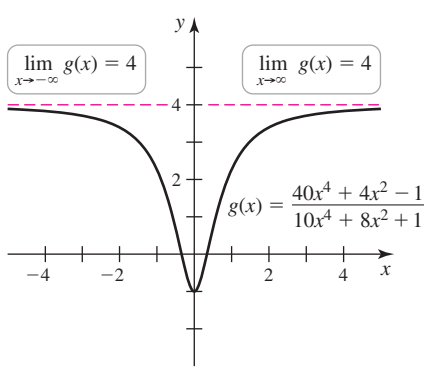


FIGURE 2.37

Using the same steps (dividing each term by x^4), it can be shown that $\displaystyle\lim_{x\to-\infty} \frac{40x^4 + 4x^2 - 1}{10x^4 + 8x^2 + 1} = 4$. This function has the horizontal asymptote $y = 4$ (Figure 2.37). Notice that the degree of the polynomial in the numerator *equals* the degree of the polynomial in the denominator.

c. We divide the numerator and denominator by the largest power of x appearing in the denominator, which is x, and then take the limit:

$$\lim_{x\to\infty}\frac{x^3-2x+1}{2x+4}=\lim_{x\to\infty}\frac{\dfrac{x^3}{x}-\dfrac{2x}{x}+\dfrac{1}{x}}{\dfrac{2x}{x}+\dfrac{4}{x}}$$ Divide numerator and denominator by x.

$$=\lim_{x\to\infty}\frac{\overbrace{x^2}^{\text{arbitrarily large}}-\overbrace{2}^{\text{constant}}+\overbrace{\frac{1}{x}}^{\text{approaches 0}}}{\underbrace{2}_{\text{constant}}+\underbrace{\frac{4}{x}}_{\text{approaches 0}}}$$ Simplify.

$$=\infty.$$ Take limits.

As $x\to\infty$, all the terms in this function either approach zero or are constant—except the x^2-term in the numerator, which becomes arbitrarily large. Therefore, the limit of the function does not exist. Using a similar analysis, we find that $\lim_{x\to-\infty}\dfrac{x^3-2x+1}{2x+4}=\infty$. These limits are not finite, and so the graph of the function has no horizontal asymptote. In this case, the degree of the polynomial in the numerator is *greater than* the degree of the polynomial in the denominator.

Related Exercises 25–34 ◀

The conclusions reached in Example 3 can be generalized for all rational functions. These results are summarized in Theorem 2.7 (Exercise 74).

THEOREM 2.7 End Behavior and Asymptotes of Rational Functions

Suppose $f(x)=\dfrac{p(x)}{q(x)}$ is a rational function, where

$$p(x)=a_mx^m+a_{m-1}x^{m-1}+\cdots+a_2x^2+a_1x+a_0 \quad\text{and}$$
$$q(x)=b_nx^n+b_{n-1}x^{n-1}+\cdots+b_2x^2+b_1x+b_0,$$

with $a_m\neq 0$ and $b_n\neq 0$.

a. Degree of numerator less than degree of denominator If $m<n$, then $\lim_{x\to\pm\infty}f(x)=0$, and $y=0$ is a horizontal asymptote of f.

b. Degree of numerator equals degree of denominator If $m=n$, then $\lim_{x\to\pm\infty}f(x)=a_m/b_n$, and $y=a_m/b_n$ is a horizontal asymptote of f.

c. Degree of numerator greater than degree of denominator If $m>n$, then $\lim_{x\to\pm\infty}f(x)=\infty$ or $-\infty$, and f has no horizontal asymptote.

d. Assuming that $f(x)$ is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeros of q.

QUICK CHECK 3 Use Theorem 2.7 to find the vertical and horizontal asymptotes of $y=\dfrac{10x}{3x-1}$. ◀

Although it isn't stated explicitly, Theorem 2.7 implies that a rational function can have at most one horizontal asymptote, and whenever there is a horizontal asymptote, $\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to-\infty} \frac{p(x)}{q(x)}$. The same cannot be said of other functions, as the next examples show.

EXAMPLE 4 **End behavior of an algebraic function** Examine the end behavior of $f(x) = \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$.

SOLUTION The square root in the denominator forces us to revise the strategy used with rational functions. First, consider the limit as $x \to \infty$. The highest power of the polynomial in the denominator is 6. However, the polynomial is under a square root, so effectively, the highest power in the denominator is $\sqrt{x^6} = x^3$. Dividing the numerator and denominator by x^3, for $x > 0$, the limit is evaluated as follows:

$$\lim_{x\to\infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}} = \lim_{x\to\infty} \frac{\frac{10x^3}{x^3} - \frac{3x^2}{x^3} + \frac{8}{x^3}}{\sqrt{\frac{25x^6}{x^6} + \frac{x^4}{x^6} + \frac{2}{x^6}}} \qquad \text{Divide by } \sqrt{x^6} = x^3.$$

$$= \lim_{x\to\infty} \frac{10 - \overbrace{\frac{3}{x}}^{\text{approaches 0}} + \overbrace{\frac{8}{x^3}}^{\text{approaches 0}}}{\sqrt{25 + \underbrace{\frac{1}{x^2}}_{\text{approaches 0}} + \underbrace{\frac{2}{x^6}}_{\text{approaches 0}}}} \qquad \text{Simplify.}$$

$$= \frac{10}{\sqrt{25}} = 2. \qquad \text{Evaluate limits.}$$

➤ Recall that

$$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0. \end{cases}$$

Therefore,

$$\sqrt{x^6} = |x^3| = \begin{cases} x^3 & \text{if } x \ge 0 \\ -x^3 & \text{if } x < 0. \end{cases}$$

Because x is negative as $x \to -\infty$, we have $\sqrt{x^6} = -x^3$.

As $x \to -\infty$, x^3 is negative, so we divide numerator and denominator by $\sqrt{x^6} = -x^3$ (which is positive):

$$\lim_{x\to-\infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}} = \lim_{x\to-\infty} \frac{\frac{10x^3}{-x^3} - \frac{3x^2}{-x^3} + \frac{8}{-x^3}}{\sqrt{\frac{25x^6}{x^6} + \frac{x^4}{x^6} + \frac{2}{x^6}}} \qquad \text{Divide by } \sqrt{x^6} = -x^3 > 0.$$

$$= \lim_{x\to-\infty} \frac{-10 + \overbrace{\frac{3}{x}}^{\text{approaches 0}} - \overbrace{\frac{8}{x^3}}^{\text{approaches 0}}}{\sqrt{25 + \underbrace{\frac{1}{x^2}}_{\text{approaches 0}} + \underbrace{\frac{2}{x^6}}_{\text{approaches 0}}}} \qquad \text{Simplify.}$$

$$= -\frac{10}{\sqrt{25}} = -2. \qquad \text{Evaluate limits.}$$

The limits reveal two asymptotes, $y = 2$ and $y = -2$. Observe that the graph crosses both horizontal asymptotes (Figure 2.38).

Related Exercises 35–38 ◀

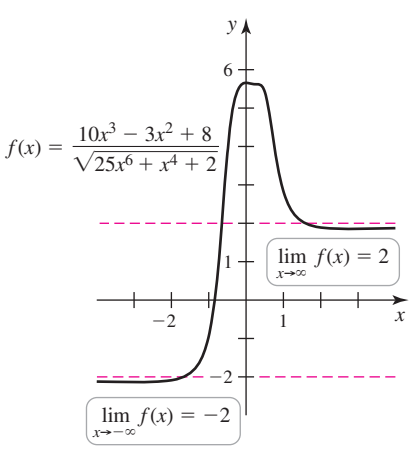


FIGURE 2.38

EXAMPLE 5 **End behavior of transcendental functions** Determine the end behavior of the following transcendental functions.

a. $f(x) = e^x$ and $g(x) = e^{-x}$ **b.** $h(x) = \ln x$ **c.** $f(x) = \cos x$

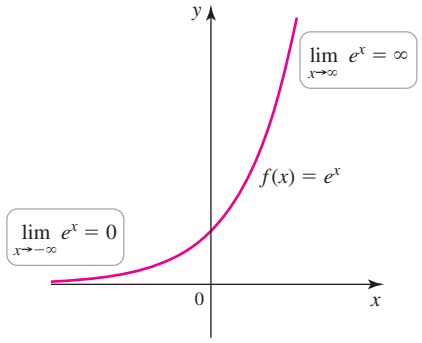


FIGURE 2.39

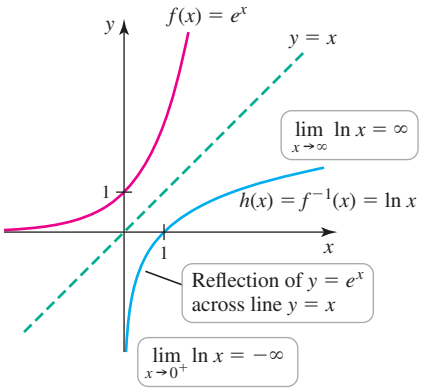


FIGURE 2.40

Table 2.9

x	$\ln x$
10	2.302
10^5	11.513
10^{10}	23.026
10^{50}	115.129
10^{99}	227.956
↓	↓
∞	???

SOLUTION

a. The graph of $f(x) = e^x$ (Figure 2.39) makes it clear that as $x \to \infty$, e^x increases without bound. All exponential functions b^x with $b > 1$ behave this way, because raising a number greater than 1 to ever-larger powers produces numbers that increase without bound. The figure also suggests that as $x \to -\infty$, the graph of e^x approaches the horizontal asymptote $y = 0$. This claim is confirmed analytically by recognizing that

$$\lim_{x\to-\infty} e^x = \lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

Therefore, $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$. Because $e^{-x} = 1/e^x$, it follows that $\lim_{x\to\infty} e^{-x} = 0$ and $\lim_{x\to-\infty} e^{-x} = \infty$.

b. The domain of $\ln x$ is $\{x: x > 0\}$, so we evaluate $\lim_{x\to 0^+} \ln x$ and $\lim_{x\to\infty} \ln x$ to determine end behavior. For the first limit, recall that $\ln x$ is the inverse of e^x (Figure 2.40), and the graph of $\ln x$ is a reflection of the graph of e^x across the line $y = x$. The horizontal asymptote $(y = 0)$ of e^x is also reflected across $y = x$, becoming a vertical asymptote $(x = 0)$ for $\ln x$. These observations imply that $\lim_{x\to 0^+} \ln x = -\infty$.

It is not obvious whether the graph of $\ln x$ approaches a horizontal asymptote or whether the function grows without bound as $x \to \infty$. Furthermore, the numerical evidence (Table 2.9) is inconclusive because $\ln x$ increases very slowly. The inverse relation between e^x and $\ln x$ is again useful. The fact that the *domain* of e^x is $(-\infty, \infty)$ implies that the *range* of $\ln x$ is also $(-\infty, \infty)$. Therefore, the values of $\ln x$ lie in the interval $(-\infty, \infty)$, and it follows that $\lim_{x\to\infty} \ln x = \infty$.

c. The cosine function oscillates between -1 and 1 as x approaches infinity (Figure 2.41). Therefore, $\lim_{x\to\infty} \cos x$ does not exist. For the same reason, $\lim_{x\to-\infty} \cos x$ does not exist.

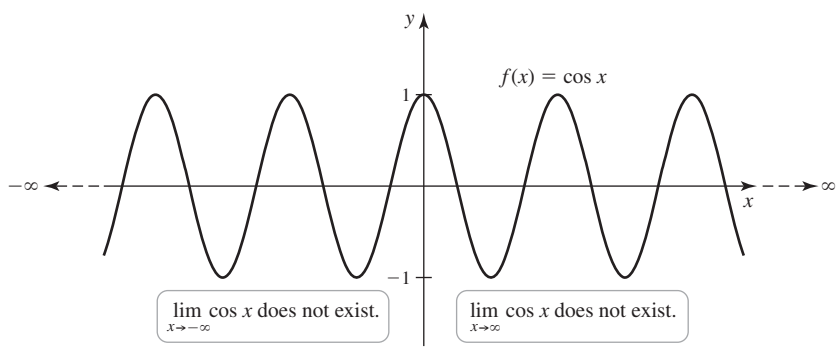


FIGURE 2.41

Related Exercises 39–44

The end behavior of exponential and logarithmic functions are important in upcoming work. We summarize these results in the following theorem.

QUICK CHECK 4 How do the functions e^{10x} and e^{-10x} behave as $x \to \infty$ and as $x \to -\infty$?

THEOREM 2.8 End Behavior of e^x, e^{-x}, and $\ln x$
The end behavior for e^x and e^{-x} on $(-\infty, \infty)$ and $\ln x$ on $(0, \infty)$ is given by the following limits:

$$\lim_{x\to\infty} e^x = \infty \quad \text{and} \quad \lim_{x\to-\infty} e^x = 0,$$

$$\lim_{x\to\infty} e^{-x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} e^{-x} = \infty,$$

$$\lim_{x\to 0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x\to\infty} \ln x = \infty.$$

SECTION 2.5 EXERCISES

Review Questions

1. Explain the meaning of $\lim_{x\to-\infty} f(x) = 10$.
2. What is a horizontal asymptote?
3. Determine $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ if $f(x) \to 100{,}000$ and $g(x) \to \infty$ as $x \to \infty$.
4. Describe the end behavior of $g(x) = e^{-2x}$.
5. Describe the end behavior of $f(x) = -2x^3$.
6. The text describes three cases that arise when examining the end behavior of a rational function $f(x) = p(x)/q(x)$. Describe the end behavior associated with each case.
7. Evaluate $\lim_{x\to\infty} e^x$, $\lim_{x\to-\infty} e^x$, and $\lim_{x\to\infty} e^{-x}$.
8. Use a sketch to find the end behavior of $f(x) = \ln x$.

Basic Skills

9–14. Limits at infinity *Evaluate the following limits.*

9. $\lim_{x\to\infty}\left(3 + \frac{10}{x^2}\right)$
10. $\lim_{x\to\infty}\left(5 + \frac{1}{x} + \frac{10}{x^2}\right)$
11. $\lim_{\theta\to\infty} \frac{\cos\theta}{\theta^2}$
12. $\lim_{x\to\infty} \frac{3 + 2x + 4x^2}{x^2}$
13. $\lim_{x\to\infty} \frac{\cos x^5}{\sqrt{x}}$
14. $\lim_{x\to-\infty}\left(5 + \frac{100}{x} + \frac{\sin^4 x^3}{x^2}\right)$

15–24. Infinite limits at infinity *Determine the following limits.*

15. $\lim_{x\to\infty} x^{12}$
16. $\lim_{x\to-\infty} 3x^{11}$
17. $\lim_{x\to\infty} x^{-6}$
18. $\lim_{x\to-\infty} x^{-11}$
19. $\lim_{x\to\infty} (3x^{12} - 9x^7)$
20. $\lim_{x\to-\infty} (3x^7 + x^2)$
21. $\lim_{x\to-\infty} (-3x^{16} + 2)$
22. $\lim_{x\to-\infty} 2x^{-8}$
23. $\lim_{x\to\infty} (-12x^{-5})$
24. $\lim_{x\to-\infty} (2x^{-8} + 4x^3)$

25–34. Rational functions *Evaluate* $\lim_{x\to\infty} f(x)$ *and* $\lim_{x\to-\infty} f(x)$ *for the following rational functions. Then give the horizontal asymptote of f (if any).*

25. $f(x) = \frac{4x}{20x + 1}$
26. $f(x) = \frac{3x^2 - 7}{x^2 + 5x}$
27. $f(x) = \frac{6x^2 - 9x + 8}{3x^2 + 2}$
28. $f(x) = \frac{4x^2 - 7}{8x^2 + 5x + 2}$
29. $f(x) = \frac{3x^3 - 7}{x^4 + 5x^2}$
30. $f(x) = \frac{x^4 + 7}{x^5 + x^2 - x}$
31. $f(x) = \frac{2x + 1}{3x^4 - 2}$
32. $f(x) = \frac{12x^8 - 3}{3x^8 - 2x^7}$
33. $f(x) = \frac{40x^5 + x^2}{16x^4 - 2x}$
34. $f(x) = \frac{-x^3 + 1}{2x + 8}$

35–38. Algebraic functions *Evaluate* $\lim_{x\to\infty} f(x)$ *and* $\lim_{x\to-\infty} f(x)$ *for the following functions. Then give the horizontal asymptote(s) of f (if any).*

35. $f(x) = \frac{4x^3 + 1}{2x^3 + \sqrt{16x^6 + 1}}$
36. $f(x) = \frac{\sqrt{x^2 + 1}}{2x + 1}$
37. $f(x) = \frac{\sqrt[3]{x^6 + 8}}{4x^2 + \sqrt{3x^4 + 1}}$
38. $f(x) = 4x\left(3x - \sqrt{9x^2 + 1}\right)$

39–44. Transcendental functions *Determine the end behavior of the following transcendental functions by evaluating appropriate limits. Then provide a simple sketch of the associated graph, showing asymptotes if they exist.*

39. $f(x) = -3e^{-x}$
40. $f(x) = 2^x$
41. $f(x) = 1 - \ln x$
42. $f(x) = |\ln x|$
43. $f(x) = \sin x$
44. $f(x) = \frac{50}{e^{2x}}$

Further Explorations

45. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
    - **a.** The graph of a function can never cross one of its horizontal asymptotes.
    - **b.** A rational function f can have both $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = \infty$.
    - **c.** The graph of any function can have at most two horizontal asymptotes.

46–55. Horizontal and vertical asymptotes

a. *Evaluate* $\lim_{x\to\infty} f(x)$ *and* $\lim_{x\to-\infty} f(x)$, *and then identify any horizontal asymptotes.*

b. *Find the vertical asymptotes. For each vertical asymptote* $x = a$, *evaluate* $\lim_{x\to a^-} f(x)$ *and* $\lim_{x\to a^+} f(x)$.

46. $f(x) = \frac{x^2 - 4x + 3}{x - 1}$
47. $f(x) = \frac{2x^3 + 10x^2 + 12x}{x^3 + 2x^2}$
48. $f(x) = \frac{\sqrt{16x^4 + 64x^2} + x^2}{2x^2 - 4}$
49. $f(x) = \frac{3x^4 + 3x^3 - 36x^2}{x^4 - 25x^2 + 144}$
50. $f(x) = 16x^2\left(4x^2 - \sqrt{16x^4 + 1}\right)$
51. $f(x) = \frac{x^2 - 9}{x(x - 3)}$
52. $f(x) = \frac{x - 1}{x^{2/3} - 1}$
53. $f(x) = \frac{\sqrt{x^2 + 2x + 6} - 3}{x - 1}$

54. $f(x) = \dfrac{|1 - x^2|}{x(x + 1)}$

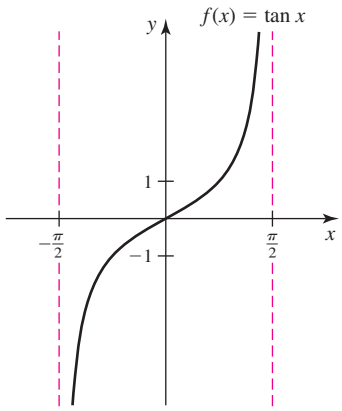
55. $f(x) = \sqrt{|x|} - \sqrt{|x - 1|}$

56–59. End behavior for transcendental functions

56. The central branch of $f(x) = \tan x$ is shown in the figure.

a. Evaluate $\lim\limits_{x \to \pi/2^-} \tan x$ and $\lim\limits_{x \to -\pi/2^+} \tan x$. Are these infinite limits or limits at infinity?

b. Sketch a graph of $g(x) = \tan^{-1} x$ by reflecting the graph of f over the line $y = x$, and use it to evaluate $\lim\limits_{x \to \infty} \tan^{-1} x$ and $\lim\limits_{x \to -\infty} \tan^{-1} x$.



T **57.** Graph $y = \sec^{-1} x$ and evaluate the following limits using the graph. Assume the domain is $\{x: |x| \geq 1\}$.

a. $\lim\limits_{x \to \infty} \sec^{-1} x$ **b.** $\lim\limits_{x \to -\infty} \sec^{-1} x$

58. The **hyperbolic cosine function**, denoted $\cosh x$, is used to model the shape of a hanging cable (a telephone wire, for example). It is defined as $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

a. Determine its end behavior by evaluating $\lim\limits_{x \to \infty} \cosh x$ and $\lim\limits_{x \to -\infty} \cosh x$.

b. Evaluate cosh 0. Use symmetry and part (a) to sketch a plausible graph for $y = \cosh x$.

59. The **hyperbolic sine function** is defined as $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

a. Determine its end behavior by evaluating $\lim\limits_{x \to \infty} \sinh x$ and $\lim\limits_{x \to -\infty} \sinh x$.

b. Evaluate sinh 0. Use symmetry and part (a) to sketch a plausible graph for $y = \sinh x$.

60–61. Sketching graphs *Sketch a possible graph of a function f that satisfies all the given conditions. Be sure to identify all vertical and horizontal asymptotes.*

60. $f(-1) = -2$, $f(1) = 2$, $f(0) = 0$, $\lim\limits_{x \to \infty} f(x) = 1$, $\lim\limits_{x \to -\infty} f(x) = -1$

61. $\lim\limits_{x \to 0^+} f(x) = \infty$, $\lim\limits_{x \to 0^-} f(x) = -\infty$, $\lim\limits_{x \to \infty} f(x) = 1$, $\lim\limits_{x \to -\infty} f(x) = -2$

62. **Asymptotes** Find the vertical and horizontal asymptotes of $f(x) = e^{1/x}$.

63. **Asymptotes** Find the vertical and horizontal asymptotes of $f(x) = \dfrac{\cos x + 2\sqrt{x}}{\sqrt{x}}$.

Applications

64–69. Steady states *If a function f represents a system that varies in time, the existence of $\lim\limits_{t \to \infty} f(t)$ means that the system reaches a steady state (or equilibrium). For the following systems, determine if a steady state exists and give the steady-state value.*

64. The population of a bacteria culture is given by $p(t) = \dfrac{2500}{t + 1}$.

65. The population of a culture of tumor cells is given by $p(t) = \dfrac{3500t}{t + 1}$.

66. The amount of drug (in milligrams) in the blood after an IV tube is inserted is $m(t) = 200(1 - 2^{-t})$.

67. The value of an investment in dollars is given by $v(t) = 1000e^{0.065t}$.

68. The population of a colony of squirrels is given by $p(t) = \dfrac{1500}{3 + 2e^{-0.1t}}$.

69. The amplitude of an oscillator is given by $a(t) = 2\left(\dfrac{t + \sin t}{t}\right)$.

70–73. Looking ahead to sequences *A sequence is an infinite, ordered list of numbers that is often defined by a function. For example, the sequence $\{2, 4, 6, 8, \ldots\}$ is specified by the function $f(n) = 2n$, where $n = 1, 2, 3, \ldots$. The limit of such a sequence is $\lim\limits_{n \to \infty} f(n)$, provided the limit exists. All the limit laws for limits at infinity may be applied to limits of sequences. Find the limit of the following sequences, or state that the limit does not exist.*

70. $\left\{4, 2, \frac{4}{3}, 1, \frac{4}{5}, \frac{2}{3}, \ldots\right\}$, which is defined by $f(n) = \dfrac{4}{n}$, for $n = 1, 2, 3, \ldots$

71. $\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$, which is defined by $f(n) = \dfrac{n - 1}{n}$, for $n = 1, 2, 3, \ldots$

72. $\left\{\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \ldots\right\}$, which is defined by $f(n) = \dfrac{n^2}{n + 1}$, for $n = 1, 2, 3, \ldots$

73. $\left\{2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \ldots\right\}$, which is defined by $f(n) = \dfrac{n + 1}{n^2}$, for $n = 1, 2, 3, \ldots$

Additional Exercises

74. End behavior of a rational function Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function, where
$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_2x^2 + a_1x + a_0$,
$q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0$, $a_m \neq 0$, and $b_n \neq 0$.

a. Prove that if $m = n$, then $\lim\limits_{x \to \pm\infty} f(x) = \dfrac{a_m}{b_n}$.

b. Prove that if $m < n$, then $\lim\limits_{x \to \pm\infty} f(x) = 0$.

T **75–76. Limits of exponentials** *Evaluate* $\lim\limits_{x \to \infty} f(x)$ *and* $\lim\limits_{x \to -\infty} f(x)$. *Then state the horizontal asymptote(s) of* f. *Confirm your findings by plotting* f.

75. $f(x) = \dfrac{2e^x + 3e^{2x}}{e^{2x} + e^{3x}}$

76. $f(x) = \dfrac{3e^x + e^{-x}}{e^x + e^{-x}}$

T **77. Subtle asymptotes** Use analytical methods to identify all the asymptotes of $f(x) = \dfrac{\ln(9 - x^2)}{2e^x - e^{-x}}$. Then confirm your results by locating the asymptotes using a graphing calculator.

QUICK CHECK ANSWERS

1. $10/11, 100/101, 1000/1001, 1$ **2.** $p(x) \to -\infty$ as $x \to \infty$ and $p(x) \to \infty$ as $x \to -\infty$ **3.** Horizontal asymptote is $y = \frac{10}{3}$; vertical asymptote is $x = \frac{1}{3}$. **4.** $\lim\limits_{x \to \infty} e^{10x} = \infty$, $\lim\limits_{x \to -\infty} e^{10x} = 0$, $\lim\limits_{x \to \infty} e^{-10x} = 0$, $\lim\limits_{x \to -\infty} e^{-10x} = \infty$ ◀

2.6 Continuity

The graphs of many functions encountered in this text contain no holes, jumps, or breaks. For example, if $L = f(t)$ represents the length of a fish t years after it is hatched, then the length of the fish changes gradually as t increases. Consequently, the graph of $L = f(t)$ contains no breaks (Figure 2.42a). Some functions, however, do contain abrupt changes in their values. Consider a parking meter that accepts only quarters and each quarter buys 15 minutes of parking. Letting $c(t)$ be the cost (in dollars) of parking for t minutes, the graph of c has breaks at integer multiples of 15 minutes (Figure 2.42b).

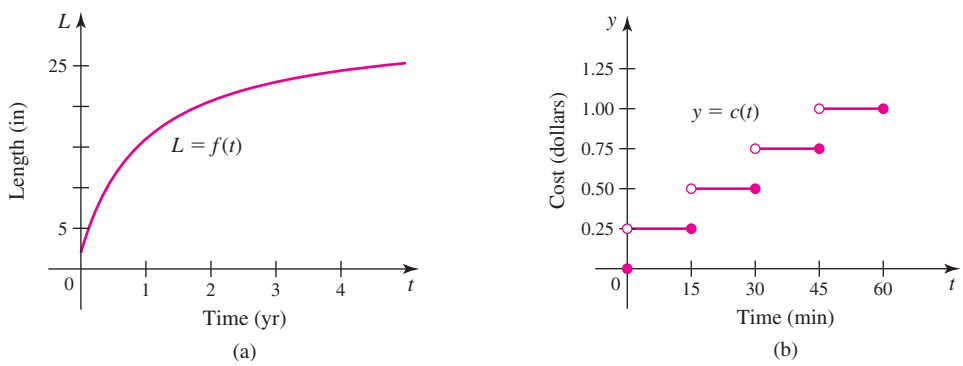


FIGURE 2.42

QUICK CHECK 1 For what values of t in $(0, 60)$ does the graph of $y = c(t)$ in Figure 2.42b have a discontinuity? ◀

Informally, we say that a function f is *continuous* at a if the graph of f contains no holes or breaks at a (that is, if the graph near a can be drawn without lifting the pencil). If a function is not continuous at a, then a is a point of discontinuity.

Continuity at a Point

This informal description of continuity is sufficient for determining the continuity of simple functions, but it is not precise enough to deal with more complicated functions such as

$$h(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

It is difficult to determine whether the graph of h has a break at 0 because it oscillates rapidly as x approaches 0 (Figure 2.43). We need a better definition.

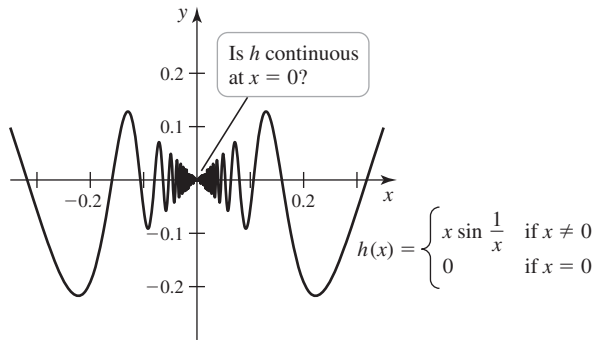


FIGURE 2.43

DEFINITION Continuity at a Point

A function f is **continuous** at a if $\lim_{x \to a} f(x) = f(a)$. If f is not continuous at a, then a is a point of discontinuity.

There is more to this definition than first appears. If $\lim_{x \to a} f(x) = f(a)$, then $f(a)$ and $\lim_{x \to a} f(x)$ must both exist, and they must be equal. The following checklist is helpful in determining whether a function is continuous at a.

Continuity Checklist

In order for f to be continuous at a, the following three conditions must hold.

1. $f(a)$ is defined (a is in the domain of f).
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$ (the value of f equals the limit of f at a).

If *any* item in the continuity checklist fails to hold, the function fails to be continuous at a. From this definition, we see that continuity has an important practical consequence:

If f is continuous at a, then $\lim_{x \to a} f(x) = f(a)$, and direct substitution may be used to evaluate $\lim_{x \to a} f(x)$.

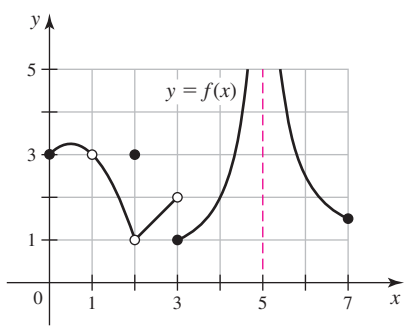


FIGURE 2.44

EXAMPLE 1 Points of discontinuity Use the graph of f in Figure 2.44 to identify values of x on the interval $(0, 7)$ at which f has a discontinuity.

SOLUTION The function f has discontinuities at $x = 1, 2, 3$, and 5 because the graph contains holes or breaks at each of these locations. These claims are verified using the continuity checklist.

- $f(1)$ is not defined.
- $f(2) = 3$ and $\lim_{x \to 2} f(x) = 1$. Therefore, $f(2)$ and $\lim_{x \to 2} f(x)$ exist but are not equal.

➤ In Example 1, the discontinuities at $x = 1$ and $x = 2$ are called **removable discontinuities** because they can be removed by redefining the function at these points (in this case $f(1) = 3$ and $f(2) = 1$). The discontinuity at $x = 3$ is called a **jump discontinuity**. The discontinuity at $x = 5$ is called an **infinite discontinuity**. These terms are discussed in Exercises 95–101.

- $\lim_{x \to 3} f(x)$ does not exist because the left-sided limit $\lim_{x \to 3^-} f(x) = 2$ differs from the right-sided limit $\lim_{x \to 3^+} f(x) = 1$.
- Neither $\lim_{x \to 5} f(x)$ nor $f(5)$ exists. *Related Exercises 9–12*◀

EXAMPLE 2 **Identifying discontinuities** Determine whether the following functions are continuous at a. Justify each answer using the continuity checklist.

a. $f(x) = \dfrac{3x^2 + 2x + 1}{x - 1}; \quad a = 1$

b. $g(x) = \dfrac{3x^2 + 2x + 1}{x - 1}; \quad a = 2$

c. $h(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}; a = 0$

SOLUTION

a. The function f is not continuous at 1 because $f(1)$ is undefined.

b. Because g is a rational function and the denominator is nonzero at 2, it follows by Theorem 2.3 that $\lim_{x \to 2} g(x) = g(2) = 17$. Therefore, g is continuous at 2.

c. By definition, $h(0) = 0$. In Exercise 55 of Section 2.3, we used the Squeeze Theorem to show that $\lim_{x \to 0} x \sin \dfrac{1}{x} = 0$. Therefore, $\lim_{x \to 0} h(x) = h(0)$, which implies that h is continuous at 0. *Related Exercises 13–20*◀

The following theorems make it easier to test various combinations of functions for continuity at a point.

THEOREM 2.9 **Continuity Rules**
If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

a. $f + g$ **b.** $f - g$
c. cf **d.** fg
e. f/g, provided $g(a) \neq 0$ **f.** $(f(x))^n$

To prove the first result, note that if f and g are continuous at a, then $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$. From the limit laws of Theorem 2.3, it follows that

$$\lim_{x \to a} (f(x) + g(x)) = f(a) + g(a).$$

Therefore, $f + g$ is continuous at a. Similar arguments lead to the continuity of differences, products, quotients, and powers of continuous functions. The next theorem is a direct consequence of Theorem 2.9.

THEOREM 2.10 Polynomial and Rational Functions

a. A polynomial function is continuous for all x.

b. A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$.

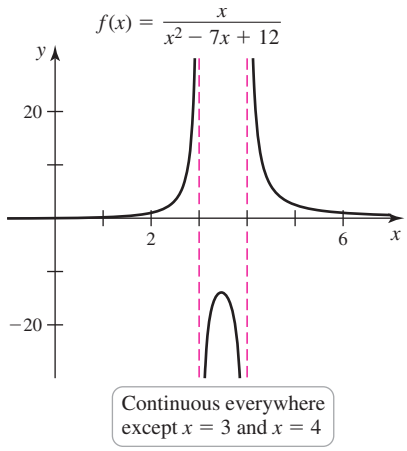


FIGURE 2.45

EXAMPLE 3 Applying the continuity theorems For what values of x is the function $f(x) = \frac{x}{x^2 - 7x + 12}$ continuous?

SOLUTION

a. Because f is rational, Theorem 2.10b implies it is continuous for all x at which the denominator is nonzero. The denominator factors as $(x - 3)(x - 4)$, so it is zero at $x = 3$ and $x = 4$. Therefore, f is continuous for all x except $x = 3$ and $x = 4$ (Figure 2.45). *Related Exercises 21–26*

The following theorem allows us to determine when a composition of two functions is continuous at a point. Its proof is informative and is outlined in Exercise 102.

THEOREM 2.11 Continuity of Composite Functions at a Point
If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

QUICK CHECK 2 Evaluate $\lim_{x \to 4} \sqrt{x^2 + 9}$ and $\sqrt{\lim_{x \to 4} (x^2 + 9)}$. How do these results illustrate that the order of a function evaluation and a limit may be switched for continuous functions?

Theorem 2.11 is useful because it allows us to conclude that the composition of two continuous functions is continuous at a point. For example, the composite function $\left(\frac{x}{x - 1}\right)^3$ is continuous for all $x \neq 1$. The theorem also says that under the stated conditions on f and g, the limit of their composition is evaluated by direct substitution; that is,

$$\lim_{x \to a} f(g(x)) = f(g(a)).$$

EXAMPLE 4 Limit of a composition Evaluate $\lim_{x \to 0} \left(\frac{x^4 - 2x + 2}{x^6 + 2x^4 + 1}\right)^{10}$.

SOLUTION The rational function $\frac{x^4 - 2x + 2}{x^6 + 2x^4 + 1}$ is continuous for all x because its denominator is always positive (Theorem 2.10b). Therefore, $\left(\frac{x^4 - 2x + 2}{x^6 + 2x^4 + 1}\right)^{10}$, which is the composition of the continuous function $f(x) = x^{10}$ and a continuous rational function, is continuous for all x by Theorem 2.11. By direct substitution,

$$\lim_{x \to 0} \left(\frac{x^4 - 2x + 2}{x^6 + 2x^4 + 1}\right)^{10} = \left(\frac{0^4 - 2 \cdot 0 + 2}{0^6 + 2 \cdot 0^4 + 1}\right)^{10} = 2^{10} = 1024.$$

Related Exercises 27–30

Closely related to Theorem 2.11 are two results dealing with limits of composite functions; they are used frequently in upcoming chapters. We present these two results—one a more general version of the other—in a single theorem.

THEOREM 2.12 Limits of Composite Functions

1. If g is continuous at a and f is continuous at $g(a)$, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

2. If $\lim_{x\to a} g(x) = L$ and f is continuous at L, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

Proof: The first statement follows directly from Theorem 2.11, which states that $\lim_{x\to a} f(g(x)) = f(g(a))$. If g is continuous at a, then $\lim_{x\to a} g(x) = g(a)$, and it follows that

$$\lim_{x\to a} f(g(x)) = f(\underbrace{g(a)}_{\lim_{x\to a} g(x)}) = f\left(\lim_{x\to a} g(x)\right).$$

The proof of the second statement relies on the formal definition of a limit, which is discussed in Section 2.7. ◀

Both statements of Theorem 2.12 justify interchanging the order of a limit and a function evaluation. By the second statement, the inner function of the composition needn't be continuous at the point of interest, but it must have a limit at that point.

EXAMPLE 5 Limits of composite functions Evaluate the following limits.

a. $\lim_{x\to -1} \sqrt{2x^2 - 1}$ **b.** $\lim_{x\to 2} \cos\left(\frac{x^2 - 4}{x - 2}\right)$

SOLUTION

a. We show later in this section that $\sqrt{x}$ is continuous for $x \geq 0$. The inner function of the composite function $\sqrt{2x^2 - 1}$ is $2x^2 - 1$ and it is continuous and positive at -1. By the first statement of Theorem 2.12,

$$\lim_{x\to -1} \sqrt{2x^2 - 1} = \sqrt{\underbrace{\lim_{x\to -1} 2x^2 - 1}_{1}} = \sqrt{1} = 1.$$

b. We show later in this section that $\cos x$ is continuous at all points of its domain. The inner function of the composite function $\cos\left(\frac{x^2 - 4}{x - 2}\right)$ is $\frac{x^2 - 4}{x - 2}$, which is not continuous at 2. However,

$$\lim_{x\to 2}\left(\frac{x^2 - 4}{x - 2}\right) = \lim_{x\to 2}\frac{(x - 2)(x + 2)}{x - 2} = \lim_{x\to 2}(x + 2) = 4.$$

Therefore, by the second statement of Theorem 2.12,

$$\lim_{x\to 2} \cos\left(\frac{x^2 - 4}{x - 2}\right) = \cos\underbrace{\left(\lim_{x\to 2}\left(\frac{x^2 - 4}{x - 2}\right)\right)}_{4} = \cos 4 \approx -0.654.$$

Related Exercises 31–34 ◀

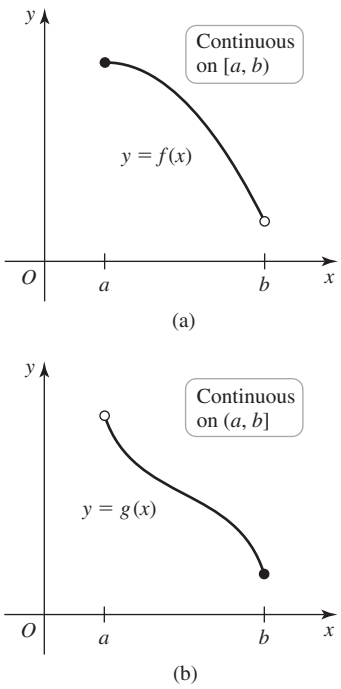


FIGURE 2.46

Continuity on an Interval

A function is *continuous on an interval* if it is continuous at every point in that interval. Consider the functions f and g whose graphs are shown in Figure 2.46. Both these functions are continuous for all x in (a, b), but what about the endpoints? To answer this question, we introduce the ideas of *left-continuity* and *right-continuity*.

DEFINITION Continuity at Endpoints

A function f is **continuous from the left** (or **left-continuous**) at a if $\lim_{x \to a^-} f(x) = f(a)$ and f is **continuous from the right** (or **right-continuous**) at a if $\lim_{x \to a^+} f(x) = f(a)$.

Combining the definitions of left-continuous and right-continuous with the definition of continuity at a point, we define what it means for a function to be continuous on an interval.

DEFINITION Continuity on an Interval

A function f is **continuous on an interval *I*** if it is continuous at all points of I. If I contains its endpoints, continuity on I means continuous from the right or left at the endpoints.

To illustrate these definitions, consider again the functions in Figure 2.46. In Figure 2.46a, f is continuous from the right at a because $\lim_{x \to a^+} f(x) = f(a)$; but it is not continuous from the left at b because $f(b)$ is not defined. Therefore, f is continuous on the interval $[a, b)$. The behavior of the function g in Figure 2.46b is the opposite: It is continuous from the left at b, but it is not continuous from the right at a. Therefore, g is continuous on $(a, b]$.

QUICK CHECK 3 Modify the graphs of the functions f and g in Figure 2.46 to obtain functions that are continuous on $[a, b]$.

EXAMPLE 6 Intervals of continuity Determine the intervals of continuity for

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \le 0 \\ 3x + 5 & \text{if } x > 0. \end{cases}$$

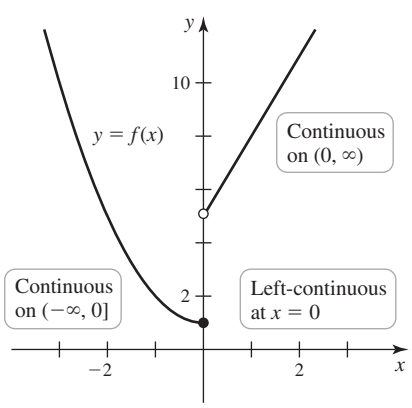


FIGURE 2.47

SOLUTION This piecewise function consists of two polynomials that describe a parabola and a line (Figure 2.47). By Theorem 2.10, f is continuous for all $x \neq 0$. From its graph, it appears that f is left-continuous at 0. This observation is verified by noting that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + 1) = 1,$$

which means that $\lim_{x \to 0^-} f(x) = f(0)$. However, because

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (3x + 5) = 5 \neq f(0),$$

we see that f is not right-continuous at 0. Therefore, we can also say that f is continuous on $(-\infty, 0]$ and on $(0, \infty)$.

Related Exercises 35–40

Functions Involving Roots

Recall that Limit Law 7 of Theorem 2.3 states

$$\lim_{x\to a} [f(x)]^{n/m} = \left[\lim_{x\to a} f(x)\right]^{n/m},$$

provided $f(x) \geq 0$, for x near a, if m is even and n/m is reduced. Therefore, if m is odd and f is continuous at a, then $[f(x)]^{n/m}$ is continuous at a, because

$$\lim_{x\to a} [f(x)]^{n/m} = \left[\lim_{x\to a} f(x)\right]^{n/m} = [f(a)]^{n/m}.$$

When m is even, the continuity of $[f(x)]^{n/m}$ must be handled more carefully because this function is defined only when $f(x) \geq 0$. Exercise 59 of Section 2.7 establishes an important fact:

If f is continuous at a and $f(a) > 0$, then f is positive for all values of x in the domain sufficiently close to a.

Combining this fact with Theorem 2.11 (the continuity of composite functions), it follows that $[f(x)]^{n/m}$ is continuous at a provided $f(a) > 0$. At points where $f(a) = 0$, the behavior of $[f(x)]^{n/m}$ varies. Often we find that $[f(x)]^{n/m}$ is left- or right-continuous at that point, or it may be continuous from both sides.

THEOREM 2.13 Continuity of Functions with Roots
Assume that m and n are positive integers with no common factors. If m is an odd integer, then $[f(x)]^{n/m}$ is continuous at all points at which f is continuous.
If m is even, then $[f(x)]^{n/m}$ is continuous at all points a at which f is continuous and $f(a) > 0$.

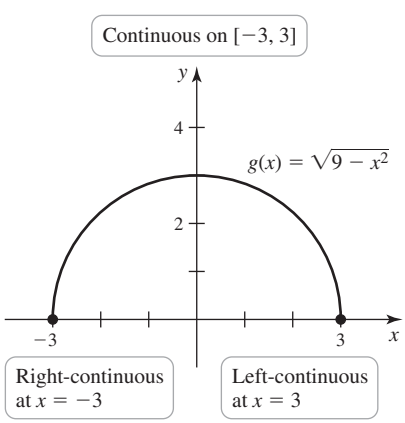


FIGURE 2.48

QUICK CHECK 4 On what interval is $f(x) = x^{1/4}$ continuous? On what interval is $f(x) = x^{2/5}$ continuous?

EXAMPLE 6 Continuity with roots For what values of x are the following functions continuous?

a. $g(x) = \sqrt{9 - x^2}$ **b.** $f(x) = (x^2 - 2x + 4)^{2/3}$

SOLUTION

a. The graph of g is the upper half of the circle $x^2 + y^2 = 9$ (which can be verified by solving $x^2 + y^2 = 9$ for y). From Figure 2.48, it appears that g is continuous on $[-3, 3]$. To verify this fact, note that g involves an even root ($m = 2, n = 1$ in Theorem 2.13). If $-3 < x < 3$, then $9 - x^2 > 0$ and by Theorem 2.13, g is continuous for all x on $(-3, 3)$.

At the right endpoint, $\lim_{x\to 3^-} \sqrt{9 - x^2} = 0 = g(3)$ by Limit Law 7, which implies that g is left-continuous at 3. Similarly, g is right-continuous at -3 because $\lim_{x\to -3^+} \sqrt{9 - x^2} = 0 = g(-3)$. Therefore, g is continuous on $[-3, 3]$.

b. The polynomial $x^2 - 2x + 4$ is continuous for all x by Theorem 2.10a. Because f involves an odd root ($m = 3, n = 2$ in Theorem 2.13), f is continuous for all x.

Related Exercises 41–50

Continuity of Transcendental Functions

The understanding of continuity that we have developed with algebraic functions may now be applied to transcendental functions.

Trigonometric Functions In Example 8 of Section 2.3, we used the Squeeze Theorem to show that $\lim_{x\to 0} \sin x = 0$ and $\lim_{x\to 0} \cos x = 1$. Because $\sin 0 = 0$ and $\cos 0 = 1$, these

limits imply that $\sin x$ and $\cos x$ are continuous at 0. The graph of $y = \sin x$ (Figure 2.49) suggests that $\lim_{x \to a} \sin x = \sin a$ for any value of a, which means that $\sin x$ is continuous everywhere. The graph of $y = \cos x$ also indicates that $\cos x$ is continuous for all x. Exercise 105 outlines a proof of these results.

With these facts in hand, we appeal to Theorem 2.9e to discover that the remaining trigonometric functions are continuous on their domains. For example, because $\sec x = 1/\cos x$, the secant function is continuous for all x for which $\cos x \neq 0$ (for all x except odd multiples of $\pi/2$) (Figure 2.50). Likewise, the tangent, cotangent, and cosecant functions are continuous at all points of their domains.

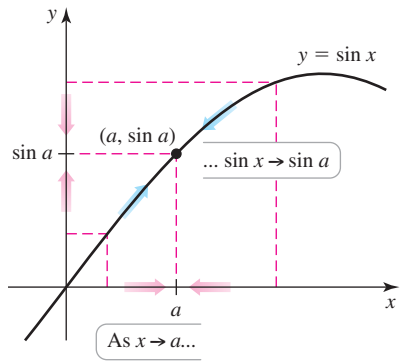


FIGURE 2.49

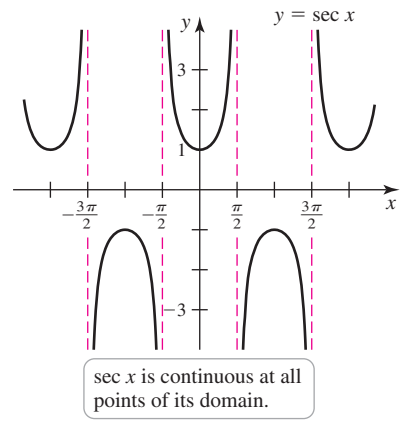


FIGURE 2.50

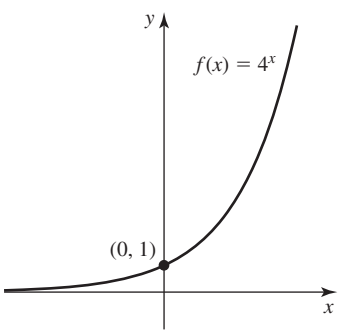


FIGURE 2.51

Exponential Functions The continuity of exponential functions of the form $f(x) = b^x$, with $0 < b < 1$ or $b > 1$, raises an important question. Consider the function $f(x) = 4^x$ (Figure 2.51). Evaluating f is routine if x is rational:

$$4^3 = 4 \cdot 4 \cdot 4 = 64; \quad 4^{-2} = \frac{1}{4^2} = \frac{1}{16}; \quad 4^{3/2} = \sqrt{4^3} = 8; \quad \text{and} \quad 4^{-1/3} = \frac{1}{\sqrt[3]{4}}.$$

But what is meant by 4^x when x is an irrational number, such as $\sqrt{2}$? In order for $f(x) = 4^x$ to be continuous for all real numbers, it must also be defined when x is an irrational number. Providing a working definition for an expression such as $4^{\sqrt{2}}$ requires mathematical results that don't appear until Chapter 6. Until then, we assume without proof that the domain of $f(x) = b^x$ is the set of all real numbers and that f is continuous at all points of its domain.

Inverse Functions Suppose a function f is continuous and one-to-one on an interval I. Reflecting the graph of f through the line $y = x$ generates the graph of f^{-1}. The reflection process introduces no discontinuities in the graph of f^{-1}, so it is plausible (and indeed, true) that f^{-1} is continuous on the interval corresponding to I. We state this fact without a formal proof.

THEOREM 2.14 Continuity of Inverse Functions
If a continuous function f has an inverse on an interval I, then its inverse f^{-1} is also continuous (on the interval consisting of the points $f(x)$, where x is in I).

Because all the trigonometric functions are continuous on their domains, they are also continuous when their domains are restricted for the purpose of defining inverse functions. Therefore, by Theorem 2.14, the inverse trigonometric functions are continuous at all points of their domains.

Logarithmic functions of the form $f(x) = \log_b x$ are continuous at all points of their domains for the same reason: They are inverses of exponential functions, which are one-to-one and continuous. Collecting all these facts together, we have the following theorem.

THEOREM 2.15 Continuity of Transcendental Functions
The following functions are continuous at all points of their domains.

Trigonometric		Inverse Trigonometric		Exponential	
$\sin x$	$\cos x$	$\sin^{-1} x$	$\cos^{-1} x$	b^x	e^x
$\tan x$	$\cot x$	$\tan^{-1} x$	$\cot^{-1} x$	**Logarithmic**	
$\sec x$	$\csc x$	$\sec^{-1} x$	$\csc^{-1} x$	$\log_b x$	$\ln x$

For each function listed in Theorem 2.15, we have $\lim_{x \to a} f(x) = f(a)$, provided a is in the domain of the function. This means that limits involving these functions may be evaluated by direct substitution at points in the domain.

EXAMPLE 7 Limits involving transcendental functions Evaluate the following limits after determining the continuity of the functions involved.

a. $\lim_{x \to 0} \dfrac{\cos^2 x - 1}{\cos x - 1}$ **b.** $\lim_{x \to 1} \left(\sqrt[4]{\ln x} + \tan^{-1} x \right)$

SOLUTION

➤ Limits like the one in Example 7a are denoted 0/0 and are known as *indeterminate forms*, to be studied further in Section 4.7.

a. Both $\cos^2 x - 1$ and $\cos x - 1$ are continuous for all x by Theorems 2.9 and 2.15. However, the ratio of these functions is continuous only when $\cos x - 1 \neq 0$, which occurs when x is not an integer multiple of 2π. Note that both the numerator and denominator of $\dfrac{\cos^2 x - 1}{\cos x - 1}$ approach 0 as $x \to 0$. To evaluate the limit, we factor and simplify:

$$\lim_{x \to 0} \frac{\cos^2 x - 1}{\cos x - 1} = \lim_{x \to 0} \frac{(\cos x - 1)(\cos x + 1)}{\cos x - 1} = \lim_{x \to 0} (\cos x + 1)$$

(where $\cos x - 1$ may be canceled because it is nonzero as x approaches 0). The limit on the right is now evaluated using direct substitution:

$$\lim_{x \to 0} (\cos x + 1) = \cos 0 + 1 = 2.$$

b. By Theorem 2.15, $\ln x$ is continuous on its domain $(0, \infty)$. However, $\ln x > 0$ only when $x > 1$, so Theorem 2.13 implies $\sqrt[4]{\ln x}$ is continuous on $(1, \infty)$. At $x = 1$, $\sqrt[4]{\ln x}$ is right-continuous (Quick Check 5). The domain of $\tan^{-1} x$ is all real numbers, and it is continuous on $(-\infty, \infty)$. Therefore, $f(x) = \sqrt[4]{\ln x} + \tan^{-1} x$ is continuous on $[1, \infty)$. Because the domain of f does not include points with $x < 1$, $\lim_{x \to 1^-} \left(\sqrt[4]{\ln x} + \tan^{-1} x \right)$ does not exist, which implies that $\lim_{x \to 1} \left(\sqrt[4]{\ln x} + \tan^{-1} x \right)$ does not exist. *Related Exercises 51–56* ◀

QUICK CHECK 5 Show that $f(x) = \sqrt[4]{\ln x}$ is right-continuous at $x = 1$. ◀

We close this section with an important theorem that has both practical and theoretical uses.

The Intermediate Value Theorem

A common problem in mathematics is finding solutions to equations of the form $f(x) = L$. Before attempting to find values of x satisfying this equation, it is worthwhile to determine whether a solution exists.

The existence of solutions is often established using a result known as the *Intermediate Value Theorem*. Given a function f and a constant L, we assume L lies between $f(a)$ and $f(b)$. The Intermediate Value Theorem says that if f is continuous on $[a, b]$, then the graph of f must cross the horizontal line $y = L$ at least once (Figure 2.52). Although this theorem is easily illustrated, its proof goes beyond the scope of this text.

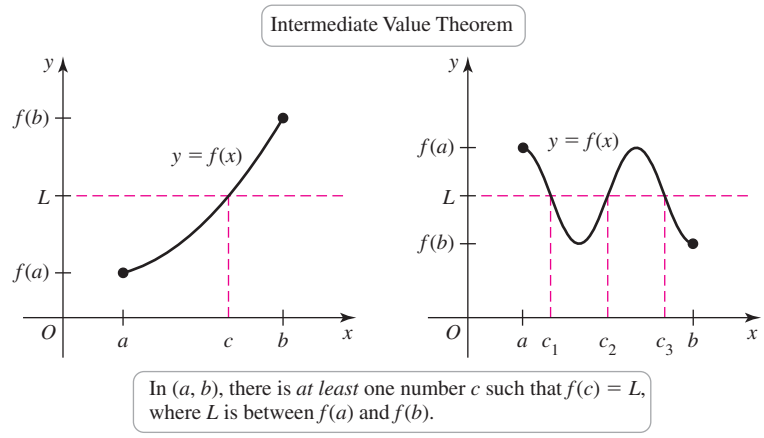


FIGURE 2.52

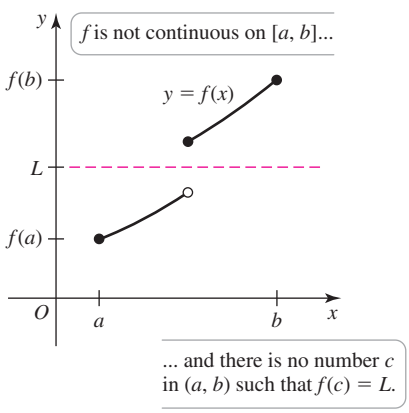


FIGURE 2.53

THEOREM 2.16 The Intermediate Value Theorem
Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.

The importance of continuity in Theorem 2.16 is illustrated in Figure 2.53, where we see a function f that is not continuous on $[a, b]$. For the value of L shown in the figure, there is no value of c in (a, b) satisfying $f(c) = L$. The next example illustrates a practical application of the Intermediate Value Theorem.

QUICK CHECK 6 Does the equation $f(x) = x^3 + x + 1 = 0$ have a solution on the interval $[-1, 1]$? Explain.

EXAMPLE 8 Finding an interest rate Suppose you invest \$1000 in a special 5-year savings account with a fixed annual interest rate r, with monthly compounding. The amount of money A in the account after 5 years (60 months) is $A(r) = 1000\left(1 + \frac{r}{12}\right)^{60}$. Your goal is to have \$1400 in the account after 5 years.

a. Use the Intermediate Value Theorem to show there is a value of r in $(0, 0.08)$—that is, an interest rate between 0% and 8%—for which $A(r) = 1400$.

b. Use a graphing utility to illustrate your explanation in part (a), and then estimate the interest rate required to reach your goal.

SOLUTION

a. As a polynomial in r (of degree 60), $A(r) = 1000\left(1 + \frac{r}{12}\right)^{60}$ is continuous for all r.
Evaluating $A(r)$ at the endpoints of the interval $[0, 0.08]$, we have $A(0) = 1000$ and $A(0.08) \approx 1489.85$. Therefore,

$$A(0) < 1400 < A(0.08),$$

and it follows, by the Intermediate Value Theorem, that there is a value of r in $(0, 0.08)$ for which $A(r) = 1400$.

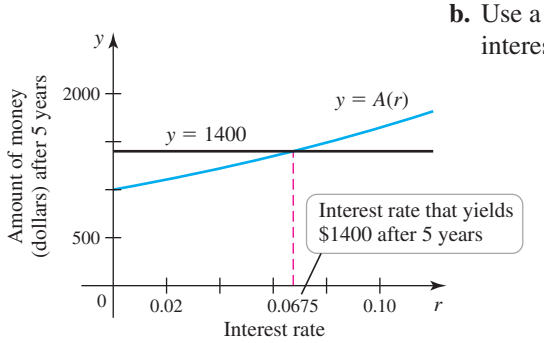


FIGURE 2.54

b. The graphs of $y = A(r)$ and the horizontal line $y = 1400$ are shown in Figure 2.54; it is evident that they intersect between $r = 0$ and $r = 0.08$. Solving $A(r) = 1400$ algebraically or using a root finder reveals that the curve and line intersect at $r \approx 0.0675$. Therefore, an interest rate of approximately 6.75% is required for the investment to be worth \$1400 after 5 years.

Related Exercises 57–64

SECTION 2.6 EXERCISES

Review Questions

1. Which of the following functions are continuous for all values in their domain? Justify your answers.

a. $a(t)$ = altitude of a skydiver t seconds after jumping from a plane

b. $n(t)$ = number of quarters needed to park in a metered parking space for t minutes

c. $T(t)$ = temperature t minutes after midnight in Chicago on January 1

d. $p(t)$ = number of points scored by a basketball player after t minutes of a basketball game

2. Give the three conditions that must be satisfied by a function to be continuous at a point.

3. What does it mean for a function to be continuous on an interval?

4. We informally describe a function f to be continuous at a if its graph contains no holes or breaks at a. Explain why this is not an adequate definition of continuity.

5. Complete the following sentences.

a. A function is continuous from the left at a if _____.

b. A function is continuous from the right at a if _____.

6. Describe the points (if any) at which a rational function fails to be continuous.

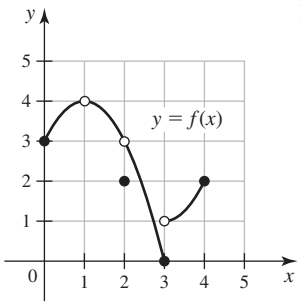
7. What is the domain of $f(x) = e^x/x$ and where is f continuous?

8. Explain in words and pictures what the Intermediate Value Theorem says.

Basic Skills

9–12. Discontinuities from a graph *Determine the points at which the following functions f have discontinuities. For each point, state the conditions in the continuity checklist that are violated.*

9.



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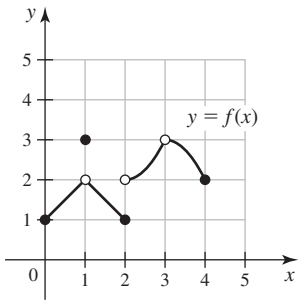

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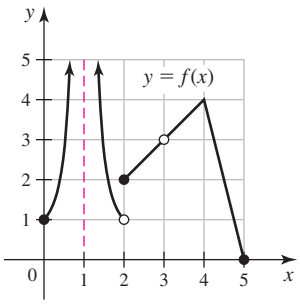

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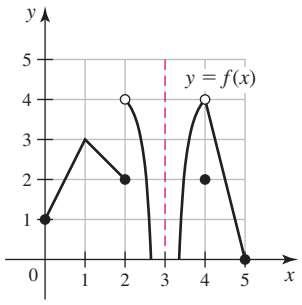


13–20. Continuity at a point *Determine whether the following functions are continuous at a. Use the continuity checklist to justify your answer.*

13. $f(x) = \dfrac{2x^2 + 3x + 1}{x^2 + 5x}$; $a = 5$

14. $f(x) = \dfrac{2x^2 + 3x + 1}{x^2 + 5x}$; $a = -5$

15. $f(x) = \sqrt{x - 2}$; $a = 1$

16. $g(x) = \dfrac{1}{x - 3}$; $a = 3$

17. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$; $a = 1$

18. $f(x) = \begin{cases} \dfrac{x^2 - 4x + 3}{x - 3} & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases}$; $a = 3$

19. $f(x) = \dfrac{5x - 2}{x^2 - 9x + 20}$; $a = 4$

20. $f(x) = \begin{cases} \dfrac{x^2 + x}{x + 1} & \text{if } x \neq -1 \\ 2 & \text{if } x = -1 \end{cases}$; $a = -1$

21–26. Continuity on intervals *Use Theorem 2.10 to determine the intervals on which the following functions are continuous.*

21. $p(x) = 4x^5 - 3x^2 + 1$

22. $g(x) = \dfrac{3x^2 - 6x + 7}{x^2 + x + 1}$

23. $f(x) = \dfrac{x^5 + 6x + 17}{x^2 - 9}$

24. $s(x) = \dfrac{x^2 - 4x + 3}{x^2 - 1}$

25. $f(x) = \dfrac{1}{x^2 - 4}$

26. $f(t) = \dfrac{t + 2}{t^2 - 4}$

27–30. Limits of compositions *Evaluate the following limits and justify your answer.*

27. $\lim\limits_{x \to 0} (x^8 - 3x^6 - 1)^{40}$

28. $\lim\limits_{x \to 2} \left(\dfrac{3}{2x^5 - 4x^2 - 50}\right)^4$

29. $\lim\limits_{x \to 1} \left(\dfrac{x + 5}{x + 2}\right)^4$

30. $\lim\limits_{x \to \infty} \left(\dfrac{2x + 1}{x}\right)^3$

31–34. Limits of composite functions *Evaluate the following limits and justify your answer.*

31. $\lim\limits_{x \to 4} \sqrt{\dfrac{x^3 - 2x^2 - 8x}{x - 4}}$

32. $\lim\limits_{t \to 4} \tan \dfrac{t - 4}{\sqrt{t} - 2}$

33. $\lim\limits_{x \to 0} \ln\left(2 \dfrac{\sin x}{x}\right)$

34. $\lim\limits_{x \to 0} \left(\dfrac{x}{\sqrt{16x + 1} - 1}\right)^{1/3}$

35–38. Intervals of continuity *Determine the intervals of continuity for the following functions.*

35. The graph of Exercise 9

36. The graph of Exercise 10

37. The graph of Exercise 11

38. The graph of Exercise 12

39. Intervals of continuity Let

$$f(x) = \begin{cases} x^2 + 3x & \text{if } x \geq 1 \\ 2x & \text{if } x < 1. \end{cases}$$

a. Use the continuity checklist to show that f is not continuous at 1.
b. Is f continuous from the left or right at 1?
c. State the interval(s) of continuity.

40. Intervals of continuity Let

$$f(x) = \begin{cases} x^3 + 4x + 1 & \text{if } x \leq 0 \\ 2x^3 & \text{if } x > 0. \end{cases}$$

a. Use the continuity checklist to show that f is not continuous at 0.
b. Is f continuous from the left or right at 0?
c. State the interval(s) of continuity.

41–46. Functions with roots *Determine the interval(s) on which the following functions are continuous. Be sure to consider right- and left-continuity at the endpoints.*

41. $f(x) = \sqrt{2x^2 - 16}$

42. $g(x) = \sqrt{x^4 - 1}$

43. $f(x) = \sqrt[3]{x^2 - 2x - 3}$

44. $f(t) = (t^2 - 1)^{3/2}$

45. $f(x) = (2x - 3)^{2/3}$

46. $f(z) = (z - 1)^{3/4}$

47–50. Limits with roots *Determine the following limits and justify your answers.*

47. $\lim\limits_{x \to 2} \sqrt{\dfrac{4x + 10}{2x - 2}}$

48. $\lim\limits_{x \to -1} \left(x^2 - 4 + \sqrt[3]{x^2 - 9}\right)$

49. $\lim\limits_{x \to 3} \left(\sqrt{x^2 + 7}\right)$

50. $\lim\limits_{t \to 2} \dfrac{t^2 + 5}{1 + \sqrt{t^2 + 5}}$

51–56. Continuity and limits with transcendental functions *Determine the interval(s) on which the following functions are continuous; then evaluate the given limits.*

51. $f(x) = \csc x$; $\lim\limits_{x \to \pi/4} f(x)$; $\lim\limits_{x \to 2\pi^-} f(x)$

52. $f(x) = e^{\sqrt{x}}$; $\lim\limits_{x \to 4} f(x)$; $\lim\limits_{x \to 0^+} f(x)$

53. $f(x) = \dfrac{1 + \sin x}{\cos x}$; $\lim\limits_{x \to \pi/2^-} f(x)$; $\lim\limits_{x \to 4\pi/3} f(x)$

54. $f(x) = \dfrac{\ln x}{\sin^{-1} x}$; $\lim\limits_{x \to 1^-} f(x)$

55. $f(x) = \dfrac{e^x}{1 - e^x}$; $\lim\limits_{x \to 0^-} f(x)$; $\lim\limits_{x \to 0^+} f(x)$

56. $f(x) = \dfrac{e^{2x} - 1}{e^x - 1}$; $\lim\limits_{x \to 0} f(x)$

T **57. Intermediate Value Theorem and interest rates** Suppose \$5000 is invested in a savings account for 10 years (120 months), with an annual interest rate of r, compounded monthly. The amount of money in the account after 10 years is $A(r) = 5000(1 + r/12)^{120}$.

a. Use the Intermediate Value Theorem to show there is a value of r in (0, 0.08)—an interest rate between 0% and 8%—that allows you to reach your savings goal of \$7000 in 10 years.
b. Use a graph to illustrate your explanation in part (a); then approximate the interest rate required to reach your goal.

T **58. Intermediate Value Theorem and mortgage payments** You are shopping for a \$150,000, 30-year (360-month) loan to buy a house. The monthly payment is

$$m(r) = \frac{150{,}000(r/12)}{1 - (1 + r/12)^{-360}},$$

where r is the annual interest rate. Suppose banks are currently offering interest rates between 6% and 8%.

a. Use the Intermediate Value Theorem to show there is a value of r in (0.06, 0.08)—an interest rate between 6% and 8%—that allows you to make monthly payments of \$1000 per month.
b. Use a graph to illustrate your explanation to part (a). Then determine the interest rate you need for monthly payments of \$1000.

T **59–64. Applying the Intermediate Value Theorem**

a. *Use the Intermediate Value Theorem to show that the following equations have a solution on the given interval.*
b. *Use a graphing utility to find all the solutions to the equation on the given interval.*
c. *Illustrate your answers with an appropriate graph.*

59. $2x^3 + x - 2 = 0$; $(-1, 1)$

60. $\sqrt{x^4 + 25x^3 + 10} = 5$; $(0, 1)$

61. $x^3 - 5x^2 + 2x = -1$; $(-1, 5)$

62. $-x^5 - 4x^2 + 2\sqrt{x} + 5 = 0$; $(0, 3)$

63. $x + e^x = 0$; $(-1, 0)$

64. $x \ln x - 1 = 0$; $(1, e)$

Further Explorations

65. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If a function is left-continuous and right-continuous at a, then it is continuous at a.
b. If a function is continuous at a, then it is left-continuous and right-continuous at a.
c. If $a < b$ and $f(a) \leq L \leq f(b)$, then there is some value of c in (a, b) for which $f(c) = L$.
d. Suppose f is continuous on $[a, b]$. Then there is a point c in (a, b) such that $f(c) = (f(a) + f(b))/2$.

66. Continuity of the absolute value function Prove that the absolute value function $|x|$ is continuous for all values of x. (*Hint:* Using the definition of the absolute value function, compute $\lim\limits_{x \to 0^-} |x|$ and $\lim\limits_{x \to 0^+} |x|$.)

67–70. Continuity of functions with absolute values *Use the continuity of the absolute value function (Exercise 66) to determine the interval(s) on which the following functions are continuous.*

67. $f(x) = |x^2 + 3x - 18|$

68. $g(x) = \left|\dfrac{x + 4}{x^2 - 4}\right|$

69. $h(x) = \left|\dfrac{1}{\sqrt{x} - 4}\right|$

70. $h(x) = |x^2 + 2x + 5| + \sqrt{x}$

71–80. Miscellaneous limits *Evaluate the following limits.*

71. $\lim\limits_{x\to\pi} \dfrac{\cos^2 x + 3\cos x + 2}{\cos x + 1}$

72. $\lim\limits_{x\to 3\pi/2} \dfrac{\sin^2 x + 6\sin x + 5}{\sin^2 x - 1}$

73. $\lim\limits_{x\to\pi/2} \dfrac{\sin x - 1}{\sqrt{\sin x} - 1}$

74. $\lim\limits_{\theta\to 0} \dfrac{\dfrac{1}{2+\sin\theta} - \dfrac{1}{2}}{\sin\theta}$

75. $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{\sin^2 x}$

76. $\lim\limits_{x\to 0^+} \dfrac{1 - \cos^2 x}{\sin x}$

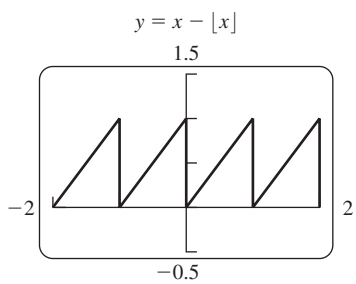
77. $\lim\limits_{x\to\infty} \dfrac{\tan^{-1} x}{x}$

78. $\lim\limits_{t\to\infty} \dfrac{\cos t}{e^{3t}}$

79. $\lim\limits_{x\to 1^-} \dfrac{x}{\ln x}$

80. $\lim\limits_{x\to 0^+} \dfrac{x}{\ln x}$

81. Pitfalls using technology The graph of the *sawtooth function* $y = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer function or floor function (Exercise 37, Section 2.2), was obtained using a graphing utility (see figure). Identify any inaccuracies appearing in the graph and then plot an accurate graph by hand.



T **82. Pitfalls using technology** Graph the function $f(x) = \dfrac{\sin x}{x}$ using a graphing window of $[-\pi, \pi] \times [0, 2]$.

a. Sketch a copy of the graph obtained with your graphing device and describe any inaccuracies appearing in the graph.

b. Sketch an accurate graph of the function. Is f continuous at 0?

c. What is the value of $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$.

83. Sketching functions

a. Sketch the graph of a function that is not continuous at 1, but is defined at 1.

b. Sketch the graph of a function that is not continuous at 1, but has a limit at 1.

84. An unknown constant Determine the value of the constant a for which the function

$$f(x) = \begin{cases} \dfrac{x^2 + 3x + 2}{x + 1} & \text{if } x \neq -1 \\ a & \text{if } x = -1 \end{cases}$$

is continuous at -1.

85. An unknown constant Let

$$g(x) = \begin{cases} x^2 + x & \text{if } x < 1 \\ a & \text{if } x = 1 \\ 3x + 5 & \text{if } x > 1. \end{cases}$$

a. Determine the value of a for which g is continuous from the left at 1.

b. Determine the value of a for which g is continuous from the right at 1.

c. Is there a value of a for which g is continuous at 1? Explain.

T **86. Asymptotes of a function containing exponentials** Let $f(x) = \dfrac{2e^x + 5e^{3x}}{e^{2x} - e^{3x}}$. Evaluate $\lim\limits_{x\to 0^-} f(x)$, $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to -\infty} f(x)$, and $\lim\limits_{x\to\infty} f(x)$. Then give the horizontal and vertical asymptotes of f. Plot f to verify your results.

T **87. Asymptotes of a function containing exponentials** Let $f(x) = \dfrac{2e^x + 10e^{-x}}{e^x + e^{-x}}$. Evaluate $\lim\limits_{x\to 0} f(x)$, $\lim\limits_{x\to -\infty} f(x)$, and $\lim\limits_{x\to\infty} f(x)$. Then give the horizontal and vertical asymptotes of f. Plot f to verify your results.

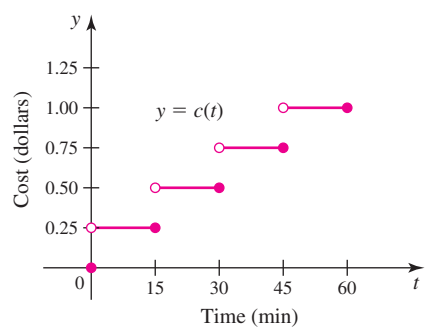
T **88–89. Applying the Intermediate Value Theorem** *Use the Intermediate Value Theorem to verify that the following equations have three solutions on the given interval. Use a graphing utility to find the approximate roots.*

88. $x^3 + 10x^2 - 100x + 50 = 0;\ (-20, 10)$

89. $70x^3 - 87x^2 + 32x - 3 = 0;\ (0, 1)$

Applications

90. Parking costs Determine the intervals of continuity for the parking cost function c introduced at the outset of this section (see figure). Consider $0 \le t \le 60$.



91. Investment problem Assume you invest \$250 at the end of each year for 10 years at an annual interest rate of r. The amount of money in your account after 10 years is $A = \dfrac{250((1+r)^{10} - 1)}{r}$.

Assume your goal is to have \$3500 in your account after 10 years.

a. Use the Intermediate Value Theorem to show that there is an interest rate r in the interval $(0.01, 0.10)$—between 1% and 10%—that allows you to reach your financial goal.

b. Use a calculator to estimate the interest rate required to reach your financial goal.

92. Applying the Intermediate Value Theorem Suppose you park your car at a trailhead in a national park and begin a 2-hr hike to a lake at 7 A.M. on a Friday morning. On Sunday morning, you leave the lake at 7 A.M. and start the 2-hr hike back to your car. Assume the lake is 3 mi from your car. Let $f(t)$ be your distance from the car t hours after 7 A.M. on Friday morning and let $g(t)$ be your distance from the car t hours after 7 A.M. on Sunday morning.

a. Evaluate $f(0), f(2), g(0)$, and $g(2)$.
b. Let $h(t) = f(t) - g(t)$. Find $h(0)$ and $h(2)$.
c. Use the Intermediate Value Theorem to show that there is some point along the trail that you will pass at exactly the same time of morning on both days.

93. The monk and the mountain A monk set out from a monastery in the valley at dawn. He walked all day up a winding path, stopping for lunch and taking a nap along the way. At dusk, he arrived at a temple on the mountaintop. The next day, the monk made the return walk to the valley, leaving the temple at dawn, walking the same path for the entire day, and arriving at the monastery in the evening. Must there be one point along the path that the monk occupied at the same time of day on both the ascent and descent? (*Hint:* The question can be answered without the Intermediate Value Theorem.) (*Source:* Arthur Koestler, *The Act of Creation.*)

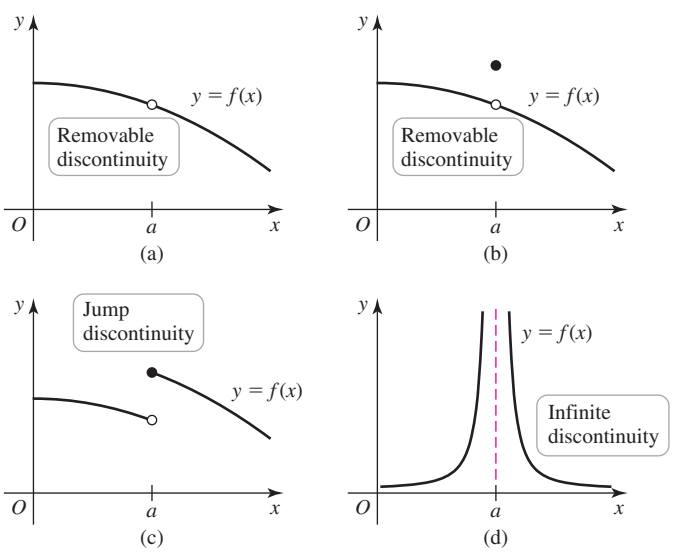
Additional Exercises

94. Does continuity of $|f|$ imply continuity of f? Let

$$g(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

a. Write a formula for $|g(x)|$.
b. Is g continuous at $x = 0$? Explain.
c. Is $|g|$ continuous at $x = 0$? Explain.
d. For any function f, if $|f|$ is continuous at a, does it necessarily follow that f is continuous at a? Explain.

95–96. Classifying discontinuities *The discontinuities in graphs (a) and (b) are* removable discontinuities *because they disappear if we define or redefine f at a so that $f(a) = \lim_{x \to a} f(x)$. The function in graph (c) has a* jump discontinuity *because left and right limits exist at a but are unequal. The discontinuity in graph (d) is an* infinite discontinuity *because the function has a vertical asymptote at a.*



95. Is the discontinuity at a in graph (c) removable? Explain.

96. Is the discontinuity at a in graph (d) removable? Explain.

97–98. Removable discontinuities *Show that the following functions have a removable discontinuity at the given point. See Exercises 95–96.*

97. $f(x) = \dfrac{x^2 - 7x + 10}{x - 2}$; $x = 2$

98. $g(x) = \begin{cases} \dfrac{x^2 - 1}{1 - x} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$; $x = 1$

99. Do removable discontinuities exist? Refer to Exercises 95–96.

a. Does the function $f(x) = x \sin(1/x)$ have a removable discontinuity at $x = 0$?
b. Does the function $g(x) = \sin(1/x)$ have a removable discontinuity at $x = 0$?

T **100–101. Classifying discontinuities** *Classify the discontinuities in the following functions at the given points. See Exercises 95–96.*

100. $f(x) = \dfrac{|x - 2|}{x - 2}$; $x = 2$

101. $h(x) = \dfrac{x^3 - 4x^2 + 4x}{x(x - 1)}$; $x = 0$ and $x = 1$

102. Continuity of composite functions Prove Theorem 2.11: If g is continuous at a and f is continuous at $g(a)$, then the composition $f \circ g$ is continuous at a. (*Hint:* Write the definition of continuity for f and g separately; then combine them to form the definition of continuity for $f \circ g$.)

103. Continuity of compositions

a. Find functions f and g such that each function is continuous at 0, but $f \circ g$ is not continuous at 0.
b. Explain why examples satisfying part (a) do not contradict Theorem 2.11.

104. Violation of the Intermediate Value Theorem? Let $f(x) = \dfrac{|x|}{x}$. Then $f(-2) = -1$ and $f(2) = 1$. Therefore, $f(-2) < 0 < f(2)$, but there is no value of c between -2 and 2 for which $f(c) = 0$. Does this fact violate the Intermediate Value Theorem? Explain.

105. Continuity of $\sin x$ and $\cos x$

a. Use the identity $\sin(a + h) = \sin a \cos h + \cos a \sin h$ with the fact that $\lim_{x \to 0} \sin x = 0$ to prove that $\lim_{x \to a} \sin x = \sin a$, thereby establishing that $\sin x$ is continuous for all x. (*Hint:* Let $h = x - a$ so that $x = a + h$ and note that $h \to 0$ as $x \to a$.)
b. Use the identity $\cos(a + h) = \cos a \cos h - \sin a \sin h$ with the fact that $\lim_{x \to 0} \cos x = 1$ to prove that $\lim_{x \to a} \cos x = \cos a$.

QUICK CHECK ANSWERS

1. $t = 15, 30, 45$ **2.** Both expressions have a value of 5, showing that $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$. **3.** Fill in the endpoints. **4.** $[0, \infty)$; $(-\infty, \infty)$ **5.** Note that $\lim_{x \to 1^+} \sqrt[4]{\ln x} = \sqrt[4]{\lim_{x \to 1^+} \ln x} = 0$ and $f(1) = \sqrt[4]{\ln 1} = 0$. Because the limit from the right and the value of the function at $x = 1$ are equal, the function is right-continuous at $x = 1$. **6.** The equation has a solution on the interval $[-1, 1]$ because f is continuous on $[-1, 1]$ and $f(-1) < 0 < f(1)$.

2.7 Precise Definitions of Limits

The limit definitions already encountered in this chapter are adequate for most elementary limits. However, some of the terminology used, such as *sufficiently close* and *arbitrarily large*, needs clarification. The goal of this section is to give limits a solid mathematical foundation by transforming the previous limit definitions into precise mathematical statements.

Moving Toward a Precise Definition

Assume the function f is defined for all x near a, except possibly at a. Recall that $\lim_{x \to a} f(x) = L$ means that $f(x)$ is arbitrarily close to L for all x sufficiently close (but not equal) to a. This limit definition is made precise by observing that the distance between $f(x)$ and L is $|f(x) - L|$ and that the distance between x and a is $|x - a|$. Therefore, we write $\lim_{x \to a} f(x) = L$ if we can make $|f(x) - L|$ arbitrarily small for any x, distinct from a, with $|x - a|$ sufficiently small. For instance, if we want $|f(x) - L|$ to be less than 0.1, then we must find a number $\delta > 0$ such that

$$|f(x) - L| < 0.1 \quad \text{whenever} \quad |x - a| < \delta \quad \text{and} \quad x \neq a.$$

If, instead, we want $|f(x) - L|$ to be less than 0.001, then we must find *another* number $\delta > 0$ such that

$$|f(x) - L| < 0.001 \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

For the limit to exist, it must be true that for *any* $\varepsilon > 0$, we can always find a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

- The phrase *for all x near a* means for all x in an open interval containing a.
- The Greek letters δ (delta) and ε (epsilon) represent small positive numbers when discussing limits.
- The two conditions $|x - a| < \delta$ and $x \neq a$ are written concisely as $0 < |x - a| < \delta$.

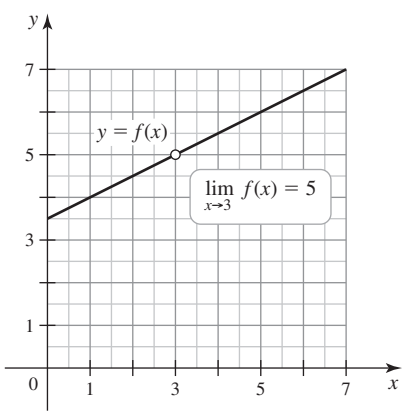


FIGURE 2.55

- The founders of calculus, Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716), developed the core ideas of calculus without using a precise definition of a limit. It was not until the 19th century that a rigorous definition was introduced by Louis Cauchy (1789–1857) and later refined by Karl Weierstrass (1815–1897).

EXAMPLE 1 Determining values of δ from a graph Figure 2.55 shows the graph of a linear function f with $\lim_{x \to 3} f(x) = 5$. For each value of $\varepsilon > 0$, determine a value of $\delta > 0$ satisfying the statement

$$|f(x) - 5| < \varepsilon \quad \text{whenever} \quad 0 < |x - 3| < \delta.$$

a. $\varepsilon = 1$

b. $\varepsilon = \frac{1}{2}$

SOLUTION

a. With $\varepsilon = 1$, we want $f(x)$ to be less than 1 unit from 5, which means $f(x)$ is between 4 and 6. To determine a corresponding value of δ, draw the horizontal lines $y = 4$ and $y = 6$ (Figure 2.56a). Then sketch vertical lines passing through the points where the horizontal lines and the graph of f intersect (Figure 2.56b). We see that the vertical lines intersect the x-axis at $x = 1$ and $x = 5$. Note that $f(x)$ is less than 1 unit from 5 on the y-axis if x is within 2 units of 3 on the x-axis. So for $\varepsilon = 1$, we let $\delta = 2$ or any smaller positive value.

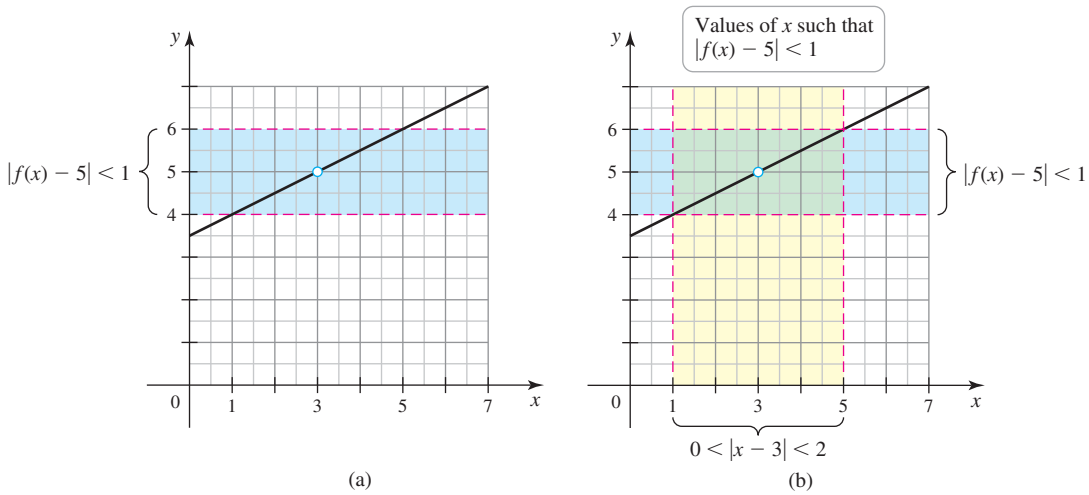


FIGURE 2.56

➤ Once an acceptable value of δ is found satisfying the statement

$$|f(x) - L| < \varepsilon \quad \text{whenever}$$
$$0 < |x - a| < \delta,$$

any smaller positive value of δ also works.

b. With $\varepsilon = \frac{1}{2}$, we want $f(x)$ to lie within a half-unit of 5 or, equivalently, $f(x)$ must lie between 4.5 and 5.5. Proceeding as in part (a), we see that $f(x)$ is within a half-unit of 5 on the y-axis (Figure 2.57a) if x is less than 1 unit from 3 (Figure 2.57b). So for $\varepsilon = \frac{1}{2}$, we let $\delta = 1$ or any smaller positive number.

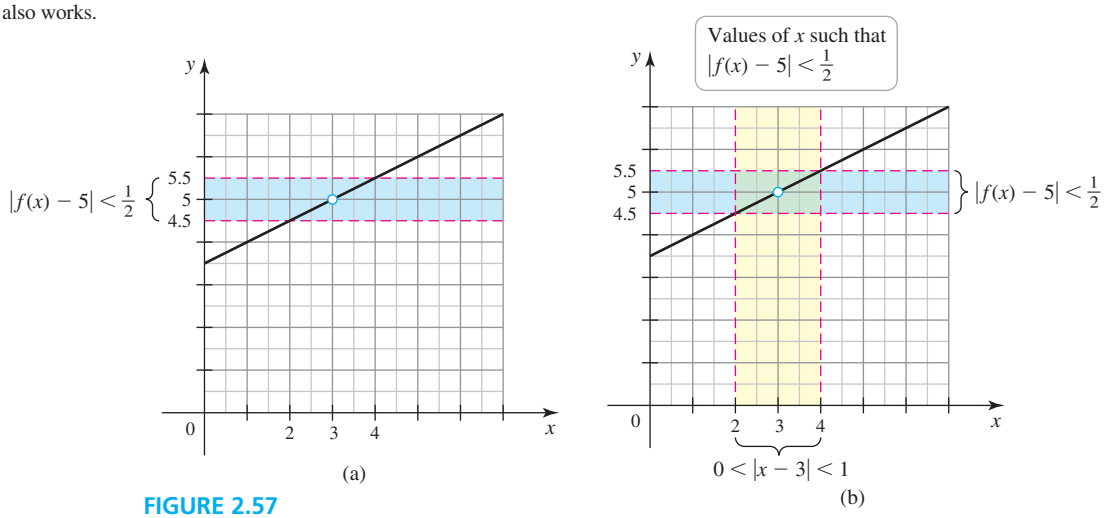


FIGURE 2.57

Related Exercises 9–12➤

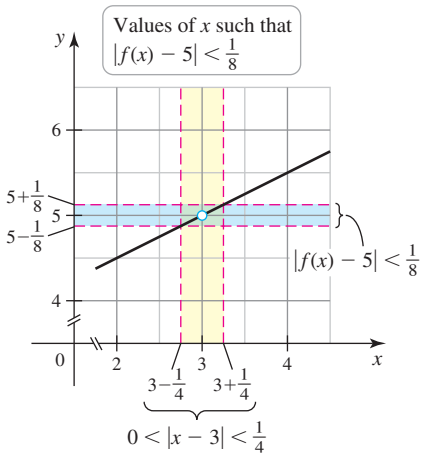


FIGURE 2.58

The idea of a limit, as illustrated in Example 1, may be described in terms of a contest between two people named Epp and Del. First, Epp picks a particular number $\varepsilon > 0$; then he challenges Del to find a corresponding value of $\delta > 0$ such that

$$|f(x) - 5| < \varepsilon \quad \text{whenever} \quad 0 < |x - 3| < \delta. \tag{1}$$

To illustrate, suppose Epp chooses $\varepsilon = 1$. From Example 1, we know that Del will satisfy (1) by choosing $0 < \delta \le 2$. If Epp chooses $\varepsilon = \frac{1}{2}$, then (by Example 1) Del responds by letting $0 < \delta \le 1$. If Epp lets $\varepsilon = \frac{1}{8}$, then Del chooses $0 < \delta \le \frac{1}{4}$ (Figure 2.58). In fact, there is a pattern: For *any* $\varepsilon > 0$ that Epp chooses, no matter how small, Del will satisfy (1) by choosing a positive value of δ satisfying $0 < \delta \le 2\varepsilon$. Del has discovered a mathematical relationship: If $0 < \delta \le 2\varepsilon$ and $0 < |x - 3| < \delta$, then $|f(x) - 5| < \varepsilon$, for *any* $\varepsilon > 0$. This conversation illustrates the general procedure for proving that $\lim_{x \to a} f(x) = L$.

QUICK CHECK 1 In Example 1, find a positive number δ satisfying the statement

$$|f(x) - 5| < \frac{1}{100} \quad \text{whenever} \quad 0 < |x - 3| < \delta.$$

A Precise Definition

Example 1 dealt with a linear function, but it points the way to a precise definition of a limit for any function. As shown in Figure 2.59, $\lim_{x \to a} f(x) = L$ means that for *any* positive number ε, there is another positive number δ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

In all limit proofs, the goal is to find a relationship between ε and δ that gives an admissible value of δ, in terms of ε only. This relationship must work for any positive value of ε.

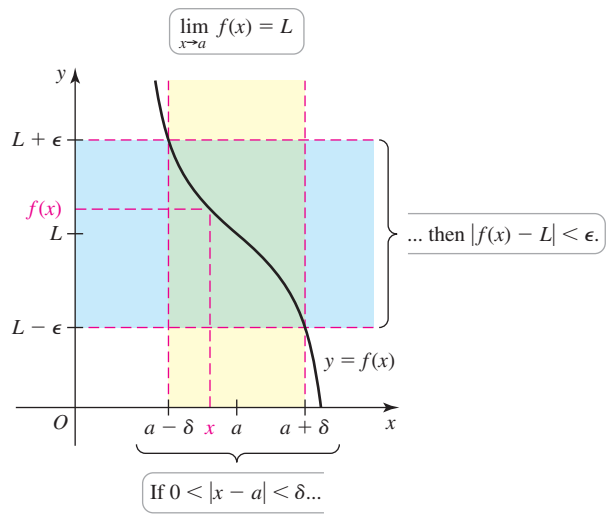


FIGURE 2.59

> The value of δ in the precise definition of a limit depends only on ε.

> Definitions of the one-sided limits $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$ are discussed in Exercises 39–43.

DEFINITION Limit of a Function

Assume that $f(x)$ exists for all x in some open interval containing a, except possibly at a. We say that the **limit of $f(x)$ as x approaches a is L**, written

$$\lim_{x \to a} f(x) = L,$$

if for *any* number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

EXAMPLE 2 Finding δ for a given ε using a graphing utility Let $f(x) = x^3 - 6x^2 + 12x - 5$ and demonstrate that $\lim_{x \to 2} f(x) = 3$ as follows. For the given values of ε, use a graphing utility to find a value of $\delta > 0$ such that

$$|f(x) - 3| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

a. $\varepsilon = 1$ **b.** $\varepsilon = \frac{1}{2}$

SOLUTION

a. The condition $|f(x) - 3| < \varepsilon = 1$ implies that $f(x)$ lies between 2 and 4. Using a graphing utility, we graph f and the lines $y = 2$ and $y = 4$ (Figure 2.60). These lines intersect the graph of f at $x = 1$ and at $x = 3$. We now sketch the vertical lines

$x = 1$ and $x = 3$ and observe that $f(x)$ is within 1 unit of 3 whenever x is within 1 unit of 2 on the x-axis (Figure 2.60). Therefore, with $\varepsilon = 1$, we can choose any δ with $0 < \delta \le 1$.

b. The condition $|f(x) - 3| < \varepsilon = \frac{1}{2}$ implies that $f(x)$ lies between 2.5 and 3.5 on the y-axis. We now find that the lines $y = 2.5$ and $y = 3.5$ intersect the graph of f at $x \approx 1.21$ and $x \approx 2.79$ (Figure 2.61). Observe that if x is less than 0.79 units from 2 on the x-axis, then $f(x)$ is less than a half-unit from 3 on the y-axis. Therefore, with $\varepsilon = \frac{1}{2}$ we can choose any δ with $0 < \delta \le 0.79$.

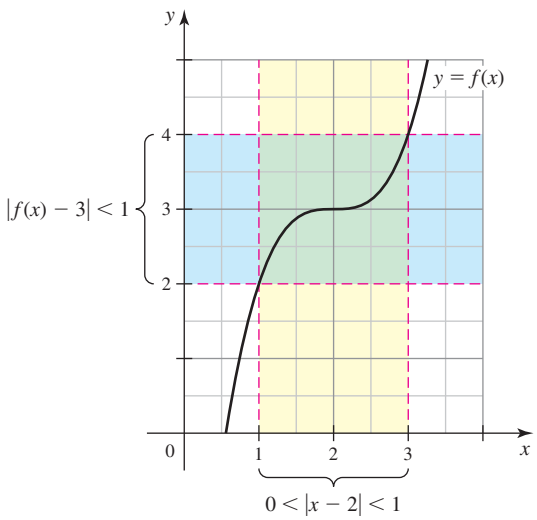


FIGURE 2.60

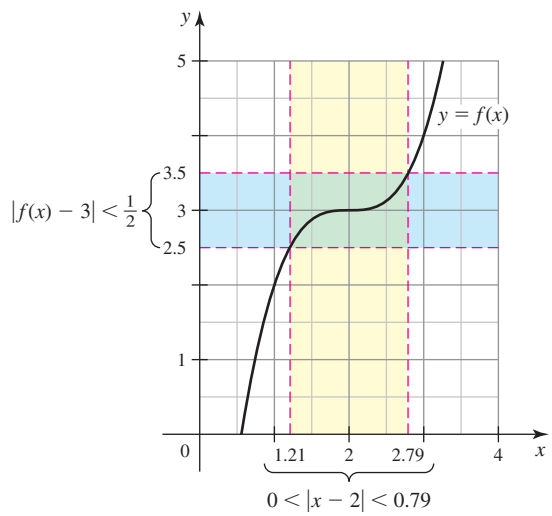


FIGURE 2.61

This procedure could be repeated for smaller and smaller values of $\varepsilon > 0$. For each value of ε, there exists a corresponding value of δ, proving that the limit exists.

Related Exercises 13–14

QUICK CHECK 2 For the function f given in Example 2, estimate a value of $\delta > 0$ satisfying $|f(x) - 3| < 0.25$ whenever $0 < |x - 2| < \delta$.

The inequality $0 < |x - a| < \delta$ means that x lies between $a - \delta$ and $a + \delta$ with $x \ne a$. We say that the interval $(a - \delta, a + \delta)$ is **symmetric about *a*** because a is the midpoint of the interval. Symmetric intervals are convenient, but Example 3 demonstrates that we don't always get symmetric intervals without a bit of extra work.

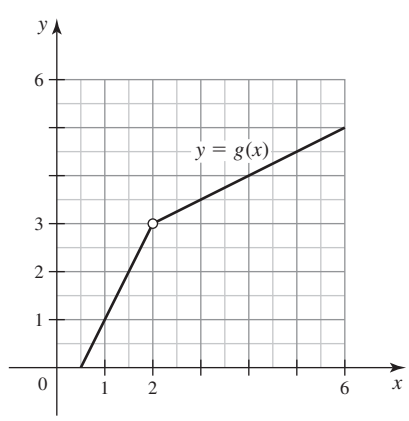


FIGURE 2.62

EXAMPLE 3 Finding a symmetric interval Figure 2.62 shows the graph of g with $\lim_{x \to 2} g(x) = 3$. For each value of ε, find the corresponding values of $\delta > 0$ that satisfy the condition

$$|g(x) - 3| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

a. $\varepsilon = 2$

b. $\varepsilon = 1$

c. For any given value of ε, make a conjecture about the corresponding values of δ that satisfy the limit condition.

SOLUTION

a. With $\varepsilon = 2$, we need a value of $\delta > 0$ such that $g(x)$ is within 2 units of 3, which means between 1 and 5, whenever x is less than δ units from 2. The horizontal lines $y = 1$ and $y = 5$ intersect the graph of g at $x = 1$ and $x = 6$. Therefore, $|g(x) - 3| < 2$ if x lies in the interval $(1, 6)$ with $x \ne 2$ (Figure 2.63a). However, we want x to lie in an interval that is *symmetric* about 2. We can guarantee that $|g(x) - 3| < 2$ only if x is less than 1 unit away from 2, on either side of 2 (Figure 2.63b). Therefore, with $\varepsilon = 2$, we take $\delta = 1$ or any smaller positive number.

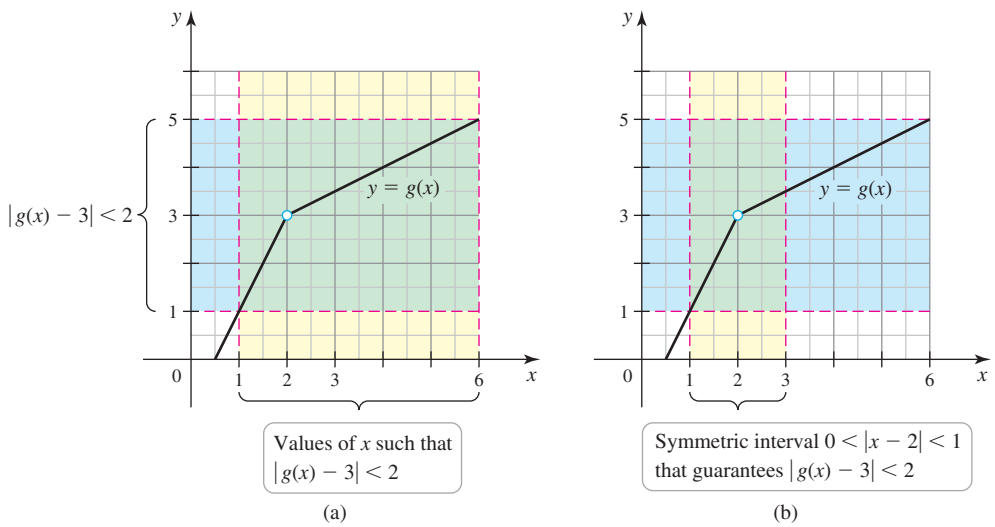


FIGURE 2.63

b. With $\varepsilon = 1$, $g(x)$ must lie between 2 and 4 (Figure 2.64a). This implies that x must be within a half-unit to the left of 2 and within 2 units to the right of 2. Therefore, $|g(x) - 3| < 1$ provided x lies in the interval $(1.5, 4)$. To obtain a symmetric interval about 2, we take $\delta = \frac{1}{2}$ or any smaller positive number. Then we are guaranteed that $|g(x) - 3| < 1$ when $0 < |x - 2| < \frac{1}{2}$ (Figure 2.64b).

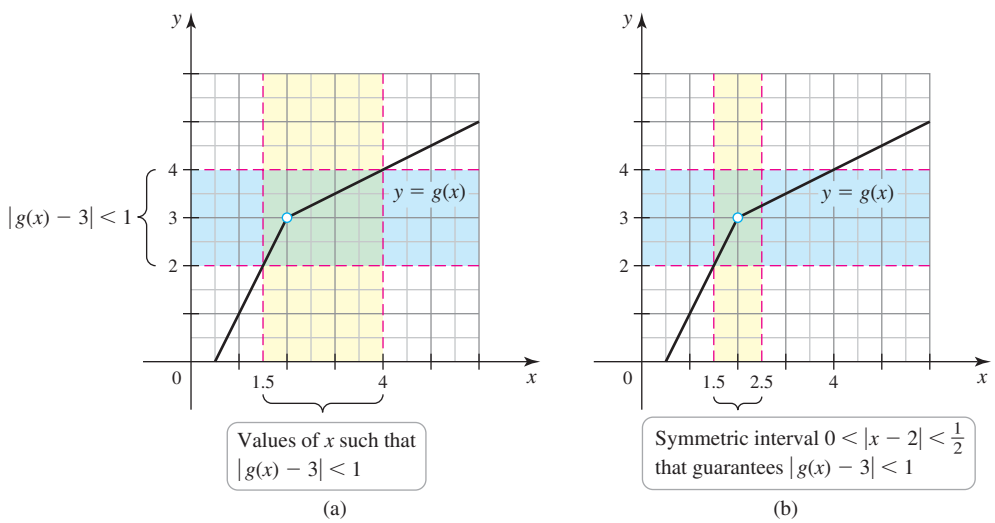


FIGURE 2.64

c. From parts (a) and (b), it appears that if we choose $\delta \le \varepsilon/2$, the limit condition is satisfied for any $\varepsilon > 0$.

Related Exercises 15–18 ◀

Limit Proofs

We use the following two-step process to prove that $\lim_{x \to a} f(x) = L$.

> The first step of the limit-proving process is the preliminary work of finding a candidate for δ. The second step verifies that the δ found in the first step actually works.

Steps for proving that $\lim_{x \to a} f(x) = L$

1. **Find δ.** Let ε be an arbitrary positive number. Use the inequality $|f(x) - L| < \varepsilon$ to find a condition of the form $|x - a| < \delta$, where δ depends only on the value of ε.
2. **Write a proof.** For any $\varepsilon > 0$, assume $0 < |x - a| < \delta$ and use the relationship between ε and δ found in Step 1 to prove that $|f(x) - L| < \varepsilon$.

EXAMPLE 4 **Limit of a linear function** Prove that $\lim_{x \to 4} (4x - 15) = 1$ using the precise definition of a limit.

SOLUTION

Step 1: *Find* δ. In this case, $a = 4$ and $L = 1$. Assuming $\varepsilon > 0$ is given, we use $|(4x - 15) - 1| < \varepsilon$ to find an inequality of the form $|x - 4| < \delta$. If $|(4x - 15) - 1| < \varepsilon$, then

$$\begin{aligned} |4x - 16| &< \varepsilon \\ 4|x - 4| &< \varepsilon \quad \text{Factor } 4x - 16. \\ |x - 4| &< \frac{\varepsilon}{4}. \quad \text{Divide by 4 and identify } \delta = \varepsilon/4. \end{aligned}$$

We have shown that $|(4x - 15) - 1| < \varepsilon$ implies $|x - 4| < \varepsilon/4$. Therefore, a plausible relationship between δ and ε is $\delta = \varepsilon/4$. We now write the actual proof.

Step 2: *Write a proof.* Let $\varepsilon > 0$ be given and assume $0 < |x - 4| < \delta$ where $\delta = \varepsilon/4$. The aim is to show that $|(4x - 15) - 1| < \varepsilon$ for all x such that $0 < |x - 4| < \delta$. We simplify $|(4x - 15) - 1|$ and isolate the $|x - 4|$ term:

$$\begin{aligned} |(4x - 15) - 1| &= |4x - 16| \\ &= 4\underbrace{|x - 4|}_{\text{less than } \delta = \varepsilon/4} \\ &< 4\left(\frac{\varepsilon}{4}\right) = \varepsilon. \end{aligned}$$

We have shown that for any $\varepsilon > 0$,

$$|f(x) - L| = |(4x - 15) - 1| < \varepsilon \quad \text{whenever} \quad 0 < |x - 4| < \delta,$$

provided $0 < \delta \le \varepsilon/4$. Therefore, $\lim_{x \to 4} (4x - 15) = 1$.

Related Exercises 19–24 ◀

Justifying Limit Laws

The precise definition of a limit is used to prove the limit laws in Theorem 2.3. Essential in several of these proofs is the triangle inequality, which states that

$$|x + y| \le |x| + |y|, \quad \text{for all real numbers } x \text{ and } y.$$

EXAMPLE 5 **Proof of Limit Law 1** Prove that if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

➤ Because $\lim_{x \to a} f(x)$ exists, if there exists a $\delta > 0$ for any given $\varepsilon > 0$, then there also exists a $\delta > 0$ for any given $\frac{\varepsilon}{2}$.

SOLUTION Assume that $\varepsilon > 0$ is given. Let $\lim_{x \to a} f(x) = L$, which implies that there exists a $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < |x - a| < \delta_1.$$

Similarly, let $\lim_{x \to a} g(x) = M$, which implies there exists a $\delta_2 > 0$ such that

$$|g(x) - M| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < |x - a| < \delta_2.$$

The minimum value of a and b is denoted $\min\{a, b\}$. If $x = \min\{a, b\}$, then x is the smaller of a and b. If $a = b$, then x equals the common value of a and b. In either case, $x \le a$ and $x \le b$.

Let $\delta = \min\{\delta_1, \delta_2\}$ and suppose $0 < |x - a| < \delta$. Because $\delta \le \delta_1$, it follows that $0 < |x - a| < \delta_1$ and $|f(x) - L| < \varepsilon/2$. Similarly, because $\delta \le \delta_2$, it follows that $0 < |x - a| < \delta_2$ and $|g(x) - M| < \varepsilon/2$. Therefore,

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| && \text{Rearrange terms.} \\ &\le |f(x) - L| + |g(x) - M| && \text{Triangle inequality.} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Proofs of other limit laws are outlined in Exercises 25 and 26.

We have shown that given any $\varepsilon > 0$, if $0 < |x - a| < \delta$, then $|(f(x) + g(x)) - (L + M)| < \varepsilon$, which implies that $\lim_{x \to a} [f(x) + g(x)] = L + M = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$. *Related Exercises 25–28*

Infinite Limits

Notice that for infinite limits, N plays the role that ε plays for regular limits. It sets a tolerance or bound for the function values $f(x)$.

In Section 2.4, we stated that $\lim_{x \to a} f(x) = \infty$ if $f(x)$ grows *arbitrarily large* as x approaches a. More precisely, this means that for any positive number N (no matter how large), $f(x)$ is larger than N if x is sufficiently close to a but not equal to a.

DEFINITION Two-Sided Infinite Limit

The **infinite limit** $\lim_{x \to a} f(x) = \infty$ means that for any positive number N, there exists a corresponding $\delta > 0$ such that

$$f(x) > N \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

As shown in Figure 2.65, to prove that $\lim_{x \to a} f(x) = \infty$, we let N represent *any* positive number. Then we find a value of $\delta > 0$, depending only on N, such that

$$f(x) > N \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

This process is similar to the two-step process for finite limits.

Precise definitions for $\lim_{x \to a} f(x) = -\infty$, $\lim_{x \to a^+} f(x) = -\infty$, $\lim_{x \to a^+} f(x) = \infty$, $\lim_{x \to a^-} f(x) = -\infty$, and $\lim_{x \to a^-} f(x) = \infty$ are given in Exercises 45–49.

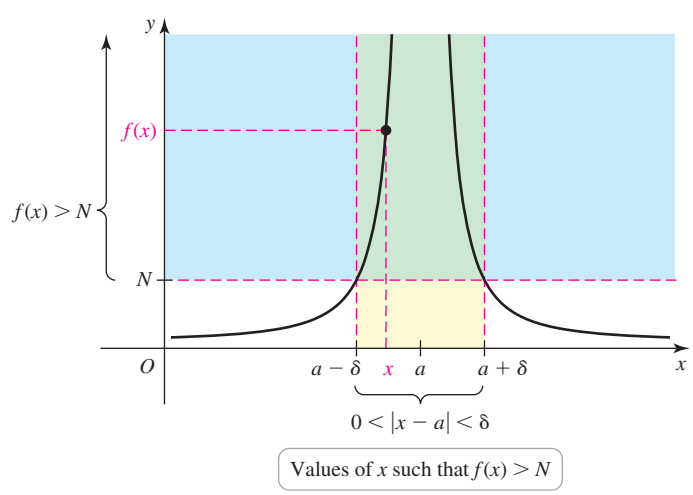


FIGURE 2.65

Steps for proving that $\lim_{x \to a} f(x) = \infty$

1. **Find δ.** Let N be an arbitrary positive number. Use the statement $f(x) > N$ to find an inequality of the form $|x - a| < \delta$, where δ depends only on N.
2. **Write a proof.** For any $N > 0$, assume $0 < |x - a| < \delta$ and use the relationship between N and δ found in Step 1 to prove that $f(x) > N$.

EXAMPLE 6 **An Infinite Limit Proof** Let $f(x) = \dfrac{1}{(x-2)^2}$. Prove that $\lim_{x \to 2} f(x) = \infty$.

SOLUTION

Step 1: *Find* $\delta > 0$. Assuming $N > 0$, we use the inequality $\dfrac{1}{(x-2)^2} > N$ to find δ, where δ depends only on N. Taking reciprocals of this inequality, it follows that

$$(x-2)^2 < \frac{1}{N}$$

$$|x-2| < \frac{1}{\sqrt{N}}. \quad \text{Take the square root of both sides.}$$

➤ Recall that $\sqrt{x^2} = |x|$.

The inequality $|x-2| < \dfrac{1}{\sqrt{N}}$ has the form $|x-2| < \delta$ if we let $\delta = \dfrac{1}{\sqrt{N}}$. We now write a proof based on this relationship between δ and N.

Step 2: *Write a proof.* Suppose $N > 0$ is given. Let $\delta = \dfrac{1}{\sqrt{N}}$ and assume $0 < |x-2| < \delta = \dfrac{1}{\sqrt{N}}$. Squaring both sides of the inequality $|x-2| < \dfrac{1}{\sqrt{N}}$ and taking reciprocals, we have

$$(x-2)^2 < \frac{1}{N} \quad \text{Square both sides.}$$

$$\frac{1}{(x-2)^2} > N. \quad \text{Take reciprocals of both sides.}$$

QUICK CHECK 3 In Example 6, if N is increased by a factor of 100, how must δ change?◀

We see that for any positive N, if $0 < |x-2| < \delta = \dfrac{1}{\sqrt{N}}$, then $f(x) = \dfrac{1}{(x-2)^2} > N$. It follows that $\lim_{x \to 2} \dfrac{1}{(x-2)^2} = \infty$. Note that because $\delta = \dfrac{1}{\sqrt{N}}$, δ decreases as N increases.

Related Exercises 29–32◀

Limits at Infinity

Precise definitions can also be written for the limits at infinity $\lim_{x \to \infty} f(x) = L$ and $\lim_{x \to -\infty} f(x) = L$. For discussion and examples, see Exercises 50 and 51.

SECTION 2.7 EXERCISES

Review Questions

1. Suppose x lies in the interval $(1, 3)$ with $x \neq 2$. Find the smallest positive value of δ such that the inequality $0 < |x - 2| < \delta$ is true.

2. Suppose $f(x)$ lies in the interval $(2, 6)$. What is the smallest value of ε such that $|f(x) - 4| < \varepsilon$?

3. Which one of the following intervals is not symmetric about $x = 5$?
   a. $(1, 9)$ **b.** $(4, 6)$ **c.** $(3, 8)$ **d.** $(4.5, 5.5)$

4. Does the set $\{x: 0 < |x - a| < \delta\}$ include the point $x = a$? Explain.

5. State the precise definition of $\lim_{x \to a} f(x) = L$.

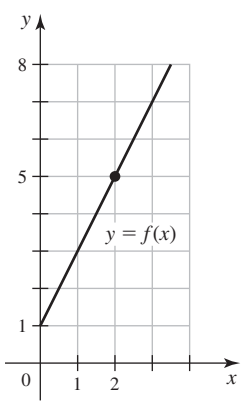
6. Interpret $|f(x) - L| < \varepsilon$ in words.

7. Suppose $|f(x) - 5| < 0.1$ whenever $0 < x < 5$. Find all values of $\delta > 0$ such that $|f(x) - 5| < 0.1$ whenever $0 < |x - 2| < \delta$.

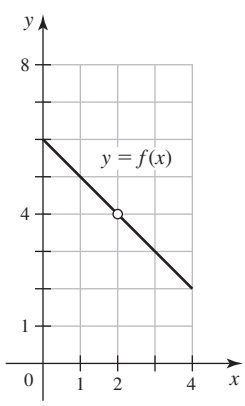
8. Give the definition of $\lim_{x \to a} f(x) = \infty$ and interpret it using pictures.

Basic Skills

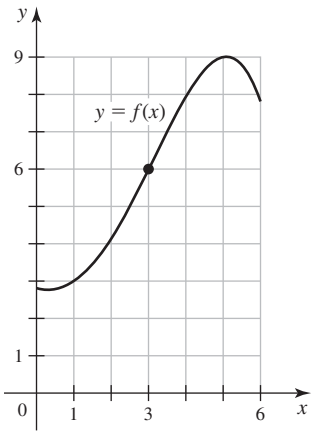
9. **Determining values of δ from a graph** The function f in the figure satisfies $\lim_{x \to 2} f(x) = 5$. Determine the largest value of $\delta > 0$ satisfying each statement.
   a. If $0 < |x - 2| < \delta$, then $|f(x) - 5| < 2$.
   b. If $0 < |x - 2| < \delta$, then $|f(x) - 5| < 1$.



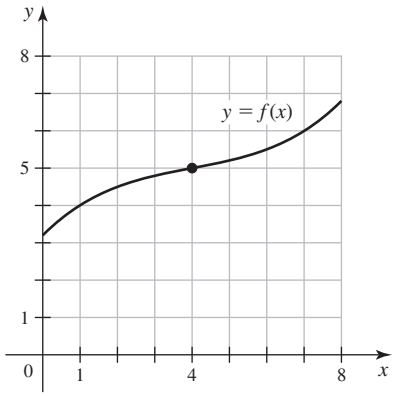
10. **Determining values of δ from a graph** The function f in the figure satisfies $\lim_{x \to 2} f(x) = 4$. Determine the largest value of $\delta > 0$ satisfying each statement.
   a. If $0 < |x - 2| < \delta$, then $|f(x) - 4| < 1$.
   b. If $0 < |x - 2| < \delta$, then $|f(x) - 4| < 1/2$.



11. **Determining values of δ from a graph** The function f in the figure satisfies $\lim_{x \to 3} f(x) = 6$. Determine the largest value of $\delta > 0$ satisfying each statement.
   a. If $0 < |x - 3| < \delta$, then $|f(x) - 6| < 3$.
   b. If $0 < |x - 3| < \delta$, then $|f(x) - 6| < 1$.



12. **Determining values of δ from a graph** The function f in the figure satisfies $\lim_{x \to 4} f(x) = 5$. Determine the largest value of $\delta > 0$ satisfying each statement.
   a. If $0 < |x - 4| < \delta$, then $|f(x) - 5| < 1$.
   b. If $0 < |x - 4| < \delta$, then $|f(x) - 5| < 0.5$.



T 13. **Finding δ for a given ε using a graph** Let $f(x) = x^3 + 3$ and note that $\lim_{x \to 0} f(x) = 3$. For each value of ε, use a graphing utility to find a value of $\delta > 0$ such that $|f(x) - 3| < \varepsilon$ whenever $0 < |x - 0| < \delta$. Sketch graphs illustrating your work.
   a. $\varepsilon = 1$ **b.** $\varepsilon = 0.5$

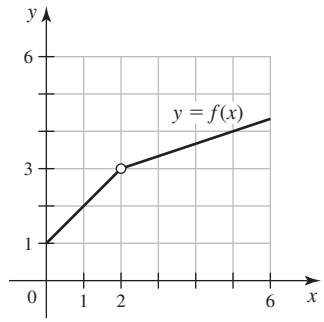
T 14. **Finding δ for a given ε using a graph** Let $g(x) = 2x^3 - 12x^2 + 26x + 4$ and note that $\lim_{x \to 2} g(x) = 24$. For each value of ε, use a graphing utility to find a value of $\delta > 0$ such that $|g(x) - 24| < \varepsilon$ whenever $0 < |x - 2| < \delta$. Sketch graphs illustrating your work.
   a. $\varepsilon = 1$ **b.** $\varepsilon = 0.5$

15. **Finding a symmetric interval** The function f in the figure satisfies $\lim_{x \to 2} f(x) = 3$. For each value of ε, find a value of $\delta > 0$ such that

$$|f(x) - 3| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta. \qquad (2)$$

   a. $\varepsilon = 1$ **b.** $\varepsilon = \frac{1}{2}$

c. For any $\varepsilon > 0$, make a conjecture about the corresponding value of δ satisfying (2).

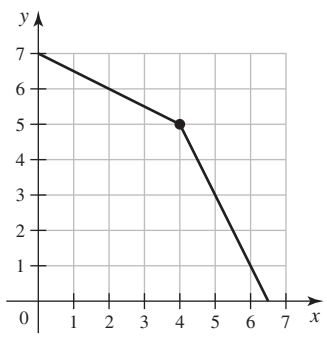


16. Finding a symmetric interval The function f in the figure satisfies $\lim_{x\to 4} f(x) = 5$. For each value of ε, find a value of $\delta > 0$ such that

$$|f(x) - 5| < \varepsilon \quad \text{whenever} \quad 0 < |x - 4| < \delta. \qquad (3)$$

a. $\varepsilon = 2$ **b.** $\varepsilon = 1$

c. For any $\varepsilon > 0$, make a conjecture about the corresponding value of δ satisfying (3).



T **17. Finding a symmetric interval** Let $f(x) = \dfrac{2x^2 - 2}{x - 1}$ and note that $\lim_{x\to 1} f(x) = 4$. For each value of ε, use a graphing utility to find a value of $\delta > 0$ such that $|f(x) - 4| < \varepsilon$ whenever $0 < |x - 1| < \delta$.

a. $\varepsilon = 2$ **b.** $\varepsilon = 1$

c. For any $\varepsilon > 0$, make a conjecture about the value of δ that satisfies the preceding inequality.

T **18. Finding a symmetric interval** Let $f(x) = \begin{cases} \frac{1}{3}x + 1 & \text{if } x \le 3 \\ \frac{1}{2}x + \frac{1}{2} & \text{if } x > 3 \end{cases}$ and note that $\lim_{x\to 3} f(x) = 2$. For each value of ε, use a graphing utility to find a value of $\delta > 0$ such that $|f(x) - 2| < \varepsilon$ whenever $0 < |x - 3| < \delta$.

a. $\varepsilon = \frac{1}{2}$ **b.** $\varepsilon = \frac{1}{4}$

c. For any $\varepsilon > 0$, make a conjecture about the value of δ that satisfies the preceding inequality.

19–24. Limit proofs *Use the precise definition of a limit to prove the following limits.*

19. $\lim_{x\to 1} (8x + 5) = 13$

20. $\lim_{x\to 3} (-2x + 8) = 2$

21. $\lim_{x\to 4} \dfrac{x^2 - 16}{x - 4} = 8$ (*Hint:* Factor and simplify.)

22. $\lim_{x\to 3} \dfrac{x^2 - 7x + 12}{x - 3} = -1$

23. $\lim_{x\to 0} x^2 = 0$ (*Hint:* Use the identity $\sqrt{x^2} = |x|$.)

24. $\lim_{x\to 3} (x - 3)^2 = 0$ (*Hint:* Use the identity $\sqrt{x^2} = |x|$.)

25. Proof of Limit Law 2 Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Prove that $\lim_{x\to a} [f(x) - g(x)] = L - M$.

26. Proof of Limit Law 3 Suppose $\lim_{x\to a} f(x) = L$. Prove that $\lim_{x\to a} [cf(x)] = cL$, where c is a constant.

27. Limit of a constant function and $f(x) = x$ Give proofs of the following theorems.

a. $\lim_{x\to a} c = c$ for any constant c

b. $\lim_{x\to a} x = a$ for any constant a

28. Continuity of linear functions Prove Theorem 2.2: If $f(x) = mx + b$, then $\lim_{x\to a} f(x) = ma + b$ for constants m and b. (*Hint:* For a given $\varepsilon > 0$, let $\delta = \varepsilon/|m|$.) Explain why this result implies that linear functions are continuous.

29–32. Limit proofs for infinite limits *Use the precise definition of infinite limits to prove the following limits.*

29. $\lim_{x\to 4} \dfrac{1}{(x - 4)^2} = \infty$

30. $\lim_{x\to -1} \dfrac{1}{(x + 1)^4} = \infty$

31. $\lim_{x\to 0} \left(\dfrac{1}{x^2} + 1\right) = \infty$

32. $\lim_{x\to 0} \left(\dfrac{1}{x^4} - \sin x\right) = \infty$

Further Explorations

33. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample. Assume a and L are finite numbers and assume $\lim_{x\to a} f(x) = L$.

a. For a given $\varepsilon > 0$, there is one value of $\delta > 0$ for which $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

b. The limit $\lim_{x\to a} f(x) = L$ means that given an arbitrary $\delta > 0$, we can always find an $\varepsilon > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

c. The limit $\lim_{x\to a} f(x) = L$ means that for any arbitrary $\varepsilon > 0$, we can always find a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

d. If $|x - a| < \delta$, then $a - \delta < x < a + \delta$.

34. Finding δ algebraically Let $f(x) = x^2 - 2x + 3$.

a. For $\varepsilon = 0.25$, find a corresponding value of $\delta > 0$ satisfying the statement

$$|f(x) - 2| < \varepsilon \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

b. Verify that $\lim_{x\to 1} f(x) = 2$ as follows. For any $\varepsilon > 0$, find a corresponding value of $\delta > 0$ satisfying the statement

$$|f(x) - 2| < \varepsilon \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

35–38. Challenging limit proofs *Use the definition of a limit to prove the following results.*

35. $\lim_{x\to 3} \dfrac{1}{x} = \dfrac{1}{3}$ (*Hint:* As $x \to 3$, eventually the distance between x and 3 will be less than 1. Start by assuming $|x - 3| < 1$ and show $\dfrac{1}{|x|} < \dfrac{1}{2}$.)

36. $\lim_{x\to 4} \dfrac{x-4}{\sqrt{x}-2} = 4$ (*Hint:* Multiply the numerator and denominator by $\sqrt{x}+2$.)

37. $\lim_{x\to 1/10} \dfrac{1}{x} = 10$ (*Hint:* To find δ, you will need to bound x away from 0. So let $\left|x - \dfrac{1}{10}\right| < \dfrac{1}{20}$.)

38. $\lim_{x\to 5} \dfrac{1}{x^2} = \dfrac{1}{25}$

39–43. Precise definitions for left- and right-sided limits *Use the following definitions.*

Assume f exists for all x near a with $x > a$. We say that the ***limit of $f(x)$ as x approaches a from the right of a is L*** *and write* $\lim_{x\to a^+} f(x) = L$, *if for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|f(x) - L| < \varepsilon \quad \textit{whenever} \quad 0 < x - a < \delta.$$

Assume f exists for all values of x near a with $x < a$. We say that the ***limit of $f(x)$ as x approaches a from the left of a is L*** *and write* $\lim_{x\to a^-} f(x) = L$, *if for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|f(x) - L| < \varepsilon \quad \textit{whenever} \quad 0 < a - x < \delta.$$

39. Comparing definitions Why is the last inequality in the definition of $\lim_{x\to a} f(x) = L$, namely, $0 < |x - a| < \delta$, replaced with $0 < x - a < \delta$ in the definition of $\lim_{x\to a^+} f(x) = L$?

40. Comparing definitions Why is the last inequality in the definition of $\lim_{x\to a} f(x) = L$, namely, $0 < |x - a| < \delta$, replaced with $0 < a - x < \delta$ in the definition of $\lim_{x\to a^-} f(x) = L$?

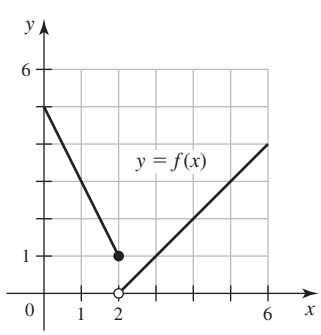
41. One-sided limit proofs Prove the following limits for

$$f(x) = \begin{cases} 3x - 4 & \text{if } x < 0 \\ 2x - 4 & \text{if } x \geq 0. \end{cases}$$

a. $\lim_{x\to 0^+} f(x) = -4$ **b.** $\lim_{x\to 0^-} f(x) = -4$
c. $\lim_{x\to 0} f(x) = -4$

42. Determining values of δ from a graph The function f in the figure satisfies $\lim_{x\to 2^+} f(x) = 0$ and $\lim_{x\to 2^-} f(x) = 1$. Determine a value of $\delta > 0$ satisfying each statement.

a. $|f(x) - 0| < 2$ whenever $0 < x - 2 < \delta$
b. $|f(x) - 0| < 1$ whenever $0 < x - 2 < \delta$
c. $|f(x) - 1| < 2$ whenever $0 < 2 - x < \delta$
d. $|f(x) - 1| < 1$ whenever $0 < 2 - x < \delta$



43. One-sided limit proof Prove that $\lim_{x\to 0^+} \sqrt{x} = 0$.

Additional Exercises

44. The relationship between one-sided and two-sided limits Prove the following statements to establish the fact that $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

a. If $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$, then $\lim_{x\to a} f(x) = L$.

b. If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

45. Definition of one-sided infinite limits We say that $\lim_{x\to a^+} f(x) = -\infty$ if for any negative number N, there exists $\delta > 0$ such that

$$f(x) < N \quad \text{whenever} \quad a < x < a + \delta.$$

a. Write an analogous formal definition for $\lim_{x\to a^+} f(x) = \infty$.

b. Write an analogous formal definition for $\lim_{x\to a^-} f(x) = -\infty$.

c. Write an analogous formal definition for $\lim_{x\to a^-} f(x) = \infty$.

46–47. One-sided infinite limits *Use the definitions given in Exercise 45 to prove the following infinite limits.*

46. $\lim_{x\to 1^+} \dfrac{1}{1-x} = -\infty$ **47.** $\lim_{x\to 1^-} \dfrac{1}{1-x} = \infty$

48–49. Definition of an infinite limit *We write* $\lim_{x\to a} f(x) = -\infty$ *if for any negative number M there exists a $\delta > 0$ such that*

$$f(x) < M \quad \textit{whenever} \quad 0 < |x - a| < \delta.$$

Use this definition to prove the following statements.

48. $\lim_{x\to 1} \dfrac{-2}{(x-1)^2} = -\infty$ **49.** $\lim_{x\to -2} \dfrac{-10}{(x+2)^4} = -\infty$

50–51. Definition of a limit at infinity *The limit at infinity* $\lim_{x\to\infty} f(x) = L$ *means that for any $\varepsilon > 0$, there exists $N > 0$ such that*

$$|f(x) - L| < \varepsilon \quad \textit{whenever} \quad x > N.$$

Use this definition to prove the following statements.

50. $\lim_{x\to\infty} \dfrac{10}{x} = 0$

51. $\lim_{x\to\infty} \dfrac{2x+1}{x} = 2$

52–53. Definition of infinite limits at infinity *We say that* $\lim_{x\to\infty} f(x) = \infty$ *if for any positive number M, there is a corresponding $N > 0$ such that*

$$f(x) > M \quad \text{whenever} \quad x > N.$$

Use this definition to prove the following statements.

52. $\lim_{x\to\infty} \dfrac{x}{100} = \infty$

53. $\lim_{x\to\infty} \dfrac{x^2+x}{x} = \infty$

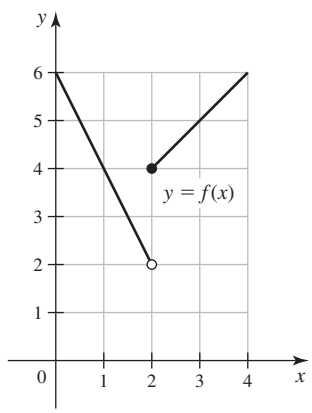
54. Proof of the Squeeze Theorem Assume the functions f, g, and h satisfy the inequality $f(x) \leq g(x) \leq h(x)$ for all values of x near a, except possibly at a. Prove that if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

55. Limit proof Suppose f is defined for all values of x near a, except possibly at a. Assume for any integer $N > 0$, there is another integer $M > 0$ such that $|f(x) - L| < 1/N$ whenever $|x - a| < 1/M$. Prove that $\lim_{x \to a} f(x) = L$ using the precise definition of a limit.

56–58. Proving that $\lim_{x \to a} f(x) \neq L$ *Use the following definition for the* nonexistence *of a limit. Assume f is defined for all values of x near a, except possibly at a. We say that $\lim_{x \to a} f(x) \neq L$ if for some $\varepsilon > 0$ there is no value of $\delta > 0$ satisfying the condition*

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

56. For the following function, note that $\lim_{x \to 2} f(x) \neq 3$. Find a value of $\varepsilon > 0$ for which the preceding condition for nonexistence is satisfied.



57. Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

58. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that $\lim_{x \to a} f(x)$ does not exist for any value of a. (*Hint:* Assume $\lim_{x \to a} f(x) = L$ for some values of a and L and let $\varepsilon = \frac{1}{2}$.)

59. A continuity proof Suppose f is continuous at a and assume $f(a) > 0$. Show that there is a positive number $\delta > 0$ for which $f(x) > 0$ for all values of x in $(a - \delta, a + \delta)$. (In other words, f is positive for all values of x in the domain sufficiently close to a.)

QUICK CHECK ANSWERS

1. $\delta = \frac{1}{50}$ or smaller **2.** $\delta = 0.62$ or smaller **3.** δ must decrease by a factor of $\sqrt{100} = 10$ (at least).

CHAPTER 2 REVIEW EXERCISES

1. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The rational function $\frac{x - 1}{x^2 - 1}$ has vertical asymptotes at $x = -1$ and $x = 1$.

b. Numerical or graphical methods always produce good estimates of $\lim_{x \to a} f(x)$.

c. The value of $\lim_{x \to a} f(x)$, if it exists, is found by calculating $f(a)$.

d. If $\lim_{x \to a} f(x) = \infty$ or $\lim_{x \to a} f(x) = -\infty$, then $\lim_{x \to a} f(x)$ does not exist.

e. If $\lim_{x \to a} f(x)$ does not exist, then either $\lim_{x \to a} f(x) = \infty$ or $\lim_{x \to a} f(x) = -\infty$.

f. If a function is continuous on the intervals (a, b) and (b, c), where $a < b < c$, then the function is also continuous on (a, c).

g. If $\lim_{x \to a} f(x)$ can be calculated by direct substitution, then f is continuous at $x = a$.

2. Estimating limits graphically Use the graph of f in the figure to find the following values, if possible.

a. $f(-1)$ **b.** $\lim_{x \to -1^-} f(x)$ **c.** $\lim_{x \to -1^+} f(x)$ **d.** $\lim_{x \to -1} f(x)$

e. $f(1)$ **f.** $\lim_{x \to 1} f(x)$ **g.** $\lim_{x \to 2} f(x)$ **h.** $\lim_{x \to 3^-} f(x)$

i. $\lim_{x \to 3^+} f(x)$ **j.** $\lim_{x \to 3} f(x)$